Intercultural Understanding, volume 11, pages 20-29

Redefinitions of Mathematical Formulae for Warping Curves Based on Three Design Methods of *Ishigaki* (Stone Walls) at Japanese Castles and Comparison with Photogrammetric Results of Edges of *Ishigaki* at Hikone Castle

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Keywords: average gradient, depth, envelope, height, initial gradient, lower bottom, occluding edge, overhang, upper bottom

Abstract: We redefine three mathematical formulae for warping curves obtained from three documents describing three design methods for the *ishigaki* (stone walls) of Japanese castles as mathematical formulae with the parameters of depth, height, and initial gradient. We also propose redefinitions of the mathematical formulae for the warping curves by selecting a lower third gradient (the average gradient of the lower third of an *ishigaki*) suitable for comparing differences in curves among the three methods and existing *ishigaki* at castles. Furthermore, the conditions under which a warping curve based on each method and an *ishigaki* designed on the basis of the methods do not overhang were compared. Finally, we compared the three redefined curves and the results of photogrammetry conducted near the edges of the *ishigaki* at Hikone Castle. The differences in the characteristics of the three types of curves and the methods were specifically clarified.

1. Introduction

Today, especially in Japan, most retaining walls are constructed with cast-in-place concrete or concrete blocks, and the number of stone walls being constructed has been decreasing. However, stone walls, which can be constructed with materials obtained from nature, have advantages such as a low impact on the natural environment, easy partial restoration, and contribution to the formation of a beautiful landscape. In fact, it has been experimentally confirmed that the strength of stone walls called *ishigaki*, which are built using the *Ano Zumi* method (a form of stone masonry by the *Ano-shu* Guild, a group of stone-wall craftspeople) is superior to that of concrete blocks, and these walls were used in part of the construction of the Shin-Meishin Expressway.

Each of the many *ishigaki* in Japan has an almost uniform gradient from the bottom to the top. However, many of the remaining castles in Japan have warping *ishigaki* with uneven gradients. These *ishigaki* are thought to have been designed to ensure structural strength by reducing the gradient at the bottom, which is subject to greater earth pressure, and to protect against enemies by increasing the gradient at the top, which is subject to less earth pressure. On the other hand, the warping *ishigaki* in castles have been recognized for their historical value due to their beauty, and they have been the subject of preservation and restoration as cultural assets. In designing modern retaining walls and stone walls, it is no longer necessary to consider security against external enemies. However, in designing stone walls that are both beautiful and structurally rational, it is extremely useful to clarify the nature of the warping curve of *ishigaki*.

The designs of the *ishigaki* at castles were considered a military secret at the time of construction, and many of them do not have drawings from that time. Therefore, the design methods became secret traditions known only to a limited number of people. Currently known once-secret documents describing the design methods for warping curves include the *Goto-ke Monjo*, reprinted by the Japan Sea Culture Research Institute, Kanazawa University (1976), the *Sekisho Sho*, reprinted by the Ishikawa Prefectural Research Institute of Kanazawa Castle (2011), and the *Ishigaki Hiden-no-sho*, reprinted by the Kyushu Research Institute for Cultural Properties (2001) and the Ishikawa Prefectural Research Institute of Kanazawa Castle. The design methods in these three documents are different, and the mathematical formulae for the warping curves for each method have already been clarified (Fujii, 2018; Morimoto, Nishida, Nishigata, & Tamano, 2000; Nishida, Nishigata, Tamano, & Morimoto, 2003; Yanai, 1988)[1]. However, because each formula defines a curve by its upper bottom, lower bottom, and height, it has the following problems. (i) Because the upper and lower bottoms cannot be measured directly on the actual *ishigaki*, it is difficult to compare them with the curves. (ii) To change the shape of the curve, two parameters, the upper and lower bottoms, need to be adjusted simultaneously, which is difficult to apply to design. (iii) Because there is no "depth" in the parameters of the defining mathematical formulae, it is difficult to vary a curve while keeping the average gradient constant and compare the dynamical stability due to the shape of the curve.

In this paper, we first redefine each mathematical formula for warping curves defined by using the three parameters of upper bottom, lower bottom, and height based on the *Goto-ke Monjo* (GM), the *Sekisho Sho* (SS), and the *Ishigaki Hiden-no-sho* (IH) using the depth and height of the warping curve and another parameter. As another parameter, we first select an initial gradient (gradient of the curve at the bottom of an *ishigaki*), which is directly obtained from each design method. We also propose redefinitions of the mathematical formulae by selecting a lower third gradient (average gradient of the lower third of an *ishigaki*) that is suitable for comparing the differences in warping curves among the three design methods and existing *ishigaki* in castles.

In addition, even though it is generally believed that *ishigaki* recede toward the top, it is also known that the curve based on the IH always has an overhang at the top (Fujii, 2018). We also clarify the conditions that lead to a lack of an overhang in the GM and SS. However, in practice, even if there is a small overhang in a curve, it can be considered not an overhang in practical use if it is smaller than the height of the stone used for the *ishigaki*. On the basis of this, the conditions under which an *ishigaki* designed by each method do not overhang in practical use are also clarified.

Finally, we compare the three redefined warping curves and the results of photogrammetry previously conducted near the edges of the *ishigaki* at Hikone Castle (Umezaki, Suzuki, Tagawa, & Yoneda, 2021). Thus, we try to investigate the mutual differences in the shape of the curves in accordance with the design methods and the differences with the existing *ishigaki*.

Through the above study, we aim to obtain the basic knowledge necessary to apply the warping curves of *ishigaki* at castles not only to the design of new stone walls but also to various architectural and landscape designs beyond stone walls.

2. Redefinitions of Mathematical Formulae of Warping Curves and Conditions for No Overhang

We will redefine the curves based on the *Goto-ke Monjo* (GM), *Sekishou Sho* (SS), and *Ishigaki Hiden-no-sho* (IH) and compare their conditions for having no overhangs.

2.1. *GOTO-KE MONJO* (GM)

The *Goto-ke Monjo* (GM), which means “Goto Family Documents,” consist of ancient documents created by the Goto Family, *ano* (*ishigaki* craftspeople) of the *Kaga-han* (Kaga Domain, based at Kanazawa Castle). Among them, the method systematized in *Yuishi Ichinin-den* (*Issatsu-bon*) in 1824 by Hikosaburo Goto (Ishikawa Prefectural Research Institute of Kanazawa Castle, 2011) is used in this study as the design method of the GM.

2.1.1. Design Method of GM

According to Kigoshi (2007; Ishikawa Prefectural Research Institute of Kanazawa Castle, 2008), Kitagaki (1987), Morimoto et al. (2000), and Nishida et al. (2003), the design method described in the GM can be formulated as follows from (a) to (f). In Figure 1, the upper end of the curve is the origin O; the x-axis heads vertically downward, and the y-axis heads horizontally outward (in front of the *ishigaki*).

(a) Determine the height $h = \mathrm{AC}$, the upper bottom $a = \mathrm{OA}$, and the lower bottom $b = \mathrm{CD}$. The point where line segment AC is divided into 2:1 is $\mathrm{B_0}$.

(b) The sectional shape of the lower third of the *ishigaki*, i.e., $\frac{2}{3}h \le x \le h$, is line DA. Let this sectional shape be line segment $\mathrm{DE_0}$, which is $\mathrm{E_0}\left(\frac{2}{3}h, \frac{2}{3}b - a\right)$, and

$$\mathrm{DE_0}:\ y = \frac{b}{h}x - a. \tag{1}$$

(c) In the top two-thirds of the *ishigaki*, divide line segment $\mathrm{B_0A}$ equally by the natural number n, and let $\mathrm{B_1}, \mathrm{B_2}, \ldots, \mathrm{B}_{n-1}$ be the points closest to $\mathrm{E_0}$ in order.

(d) Let a_0 be the value obtained by dividing a by the sum of the natural numbers from 1 to n,

$$a_0 = \frac{2a}{n(n+1)}. \tag{2}$$

(e) The point that protrudes a_0 from the intersection of the straight line drawn horizontally from $\mathrm{B_1}$ in the direction of the y-axis and straight line $\mathrm{DE_0}$ to the front of the *ishigaki* is $\mathrm{E_1}\left(\frac{2}{3}h\left(1-\frac{1}{n}\right), \frac{2}{3}b\left(1-\frac{1}{n}\right) + a_0 - a\right)$. The sectional shape of the part with $\frac{2(n-1)}{3n}h \le x \le \frac{2}{3}h$ is

$$\mathrm{E_0E_1}:\ y = \frac{1}{h}\left(b - \frac{3a}{n+1}\right)x + \frac{2a}{n+1} - a. \tag{3}$$

(f) Similarly, when $2 \le k \le n-1$, the point that protrudes a_0 from the intersection of the straight line drawn horizontally from B_k in the direction of the y-axis and straight line $\mathrm{E}_{k-2}\mathrm{E}_{k-1}$ to the front of the *ishigaki* can be obtained inductively as $\mathrm{E}_k\left(\frac{2}{3}h\left(1-\frac{k}{n}\right), \frac{2}{3}b\left(1-\frac{k}{n}\right) + \frac{k(k+1)}{2} - a\right)$. The sectional shape of the part with $\frac{2(n-k)}{3n}h \le x \le \frac{2(n-k+1)}{3n}$ is

$$\mathrm{E}_{k-1}\mathrm{E}_k: y = \frac{1}{h}\left(b - 3a\frac{k}{n+1}\right)x + 2a\frac{k}{n+1}\left(1 - \frac{k-1}{2n}\right) - a. \tag{4}$$

If $\mathrm{O} = \mathrm{E}_n$, (4) holds for $1 \le k \le n$. Therefore, the sectional shape $\mathrm{DE_0E_1} \cdots \mathrm{E}_{n-1}\mathrm{O}$ can be determined by (1) and (4).

2.1.2. Derivation of Warping Curve Based on Design Method of GM

Here, by finding the envelope of the sectional shape $\mathrm{E_0E_1} \cdots \mathrm{E}_{n-1}\mathrm{O}$, we obtain a warping curve that approximates the sectional shape between $\mathrm{E_0}$ and O when $n \to \infty$. In (4), if we replace $\frac{k}{n}$ with variable t and let $n \to \infty$, we have

$$y = \frac{1}{h}(b - 3at)x + 2at\left(1 - \frac{t}{2}\right) - a. \tag{5}$$

Multiplying both sides by h and transposing so that the right side is zero, we have

$$hy - (b - 3at)x + 2ah\left(t - \frac{t^2}{2}\right) - ah = 0. \tag{6}$$

Partial differentiation of both sides by t yields

$$3ax + 2ah(1-t) = 0. \tag{7}$$

If we find t,

$$t = 1 - \frac{3}{2h}x. \tag{8}$$

Substituting this into (5), we have

$$y = a\left(\frac{3}{2h}x - 1\right)^2 + \frac{b}{h}x - a$$
$$= \frac{4a}{9h^2}\left\{x - \frac{3}{2}h\left(1 - \frac{b}{3a}\right)\right\}^2 + \frac{2}{3}b - \frac{1}{9a}b^2 - a. \tag{9}$$

This envelope passes through two points, $\mathrm{E_0}$ and O. From the above, the sectional shape between $\mathrm{E_0}$ and O can be approximated by a quadratic function (Figure 1). Yanai (1988) derived this quadratic function with the origin at point A. It is the same mathematical formula as that in (9) with the $-a$ term removed, and it is the same curve. The same is true for the quadratic function shown by Morimoto et al. (2000) and Nishida et al. (2003)[2]. Let this be the warping curve between $\mathrm{E_0}$ and O.

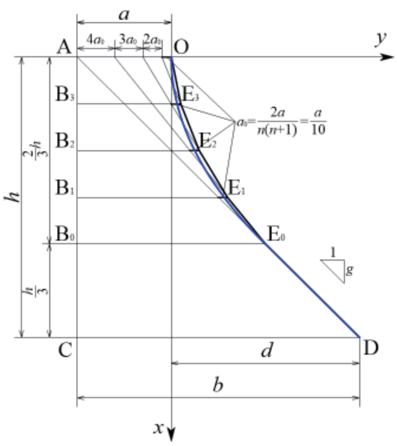


Figure 1. Design method for sectional shape based on *Goto-ke Monjo* (GM, for $n = 4$) and warping curve for $n \to \infty$ (blue)

2.1.3. Redefinition of Mathematical Formula for

Warping Curve of GM

If we rewrite (9) using the depth $d = b - a$ and the gradient of line DE_0 $g = \frac{h}{b}$ (initial gradient) instead of a and b, we have

$$y = \left(\frac{h}{g} - d\right)\left(\frac{3}{2h}x - 1\right)^2 + \frac{1}{g}(x - h) + d$$
$$= \frac{4}{9h}\left(\frac{1}{g} - \frac{d}{h}\right)\left[x - \frac{3}{2}h\left\{1 - \frac{h}{3(h - dg)}\right\}\right]^2 - \frac{h}{3g}\left\{1 + \frac{h}{3(h - dg)}\right\} + d. \quad (10)$$

2.1.4. Conditions for No Overhang for Warping Curve of GM

If the x-coordinate of the vertex of the quadratic function calculated in (9) and (10) is positive, the top of the warping curve overhangs. To prevent the top of the warping curve from overhanging, the x-coordinate of the vertex must be zero or negative. According to (9), the condition for no overhang is

$$\frac{3}{2}h\left(1 - \frac{b}{3a}\right) \leq 0. \quad (11)$$

Because a, b, and h are positive, we have

$$b \geq 3a. \quad (12)$$

Also, if we rewrite the condition in (12) using the initial gradient g instead of the upper and lower bottoms $a = \frac{h}{g} - d$ and $b = \frac{h}{g}$, because a is not positive when $g \geq \frac{h}{d}$,

$$\frac{2h}{3d} \leq g < \frac{h}{d}. \quad (13)$$

Therefore, the initial gradient of GM must be greater than or equal to 2/3 of the average gradient of the entire *ishigaki* to prevent overhang.

2.1.5. Practical Conditions for No Overhang for Ishigaki by GM

In practical use, n in the GM is a finite natural number, and there is no problem if section $E_{n-1}O$ of the topmost *ishigaki* does not overhang (Figure 2). In other words, the slope of line $E_{n-1}E_n$ should be greater than or equal to zero. Therefore, according to (4),

$$\frac{1}{h}\left(b - 3a\frac{n}{n+1}\right) \geq 0. \quad (14)$$

Therefore, if we divide line segment AB_0 into n equal parts, we need to satisfy the condition

$$b \geq 3a\left(1 - \frac{1}{n+1}\right). \quad (15)$$

If we rewrite this condition with d, h, and g, we have

$$\frac{h}{3d}\left(2 - \frac{1}{n}\right) \leq g < \frac{h}{d}. \quad (16)$$

If $n \to \infty$, (15) and (16) coincide with (12) and (13), respectively.

2.2. *SEKISHO SHO* (SS)

The *Sekisho Sho* (SS), which means "Stone Walls Document," is a document compiled in 1755 by the Yuasa Family, *ano* of the *Iwakuni-han* (Iwakini Domain, based at Iwakuni Castle).

2.2.1. Design Method by SS

The SS does not contain a general design method, but it does contain the gradient values for each *ken* (one *ken* is about 1.82 m) from the bottom for each height of the *ishigaki*. Nishida et al. (2003) proposed a design method that agrees with these gradient values with very high accuracy. This design method can be formulated as follows from (a) to (f). In Figure 3, as in Figure 1, the upper end of the curve is the origin O, the x-axis is downward vertical, and the y-axis is outward horizontal (in front of the *ishigaki*).

(a) Determine the height $h = \mathrm{AC}$, the upper bottom $a = \mathrm{OA}$, and the lower bottom $b = \mathrm{CD}$.

(b) Divide line segment AC into equal parts by natural number $n + 1$, and let $B_0, B_1, \ldots, B_{n-1}$ be the points closest to C in order.

(c) The sectional shape of part n+1 from the bottom, i.e., $\frac{n}{n+1}h \leq x \leq h$, is line DA. Let this sectional shape be line segment DE_0, which is $E_0\left(\frac{n}{n+1}h, \frac{n}{n+1}b - a\right)$, and

$$DE_0\colon\ y = \frac{b}{h}x - a. \quad (17)$$

(d) At $i = 1,2,\cdots,n$, we define a_i by

$$a_i = \frac{6ai}{n(n+1)(n+2)}. \quad (18)$$

(e) The point that protrudes a_1 from the intersection of the straight line drawn horizontally from B_1 in the direction of the y-axis and straight line DE_0 to the front of the *ishigaki* is $E_1\left(\frac{n-1}{n+1}h, \frac{n-1}{n+1}b + a_1 - a\right)$. The sectional shape of the part with $\frac{n-1}{n+1}h \leq x \leq \frac{n}{n+1}h$ is

$$E_0E_1\colon y = \frac{1}{h}\left\{b - \frac{6a}{n(n+2)}\right\}x + \frac{6a}{(n+1)(n+2)} - a. \quad (19)$$

(f) Similarly, when $2 \leq k \leq n-1$, the point that protrudes a_k from the intersection of the straight line drawn horizontally from B_k in the direction of the y-axis and straight line $E_{k-2}E_{k-1}$ to the front of the *ishigaki* can be obtained inductively as $E_k\left(\frac{n-k}{n+1}h, \frac{n-k}{n+1}b + \sum_{i=1}^{k}(k - i + 1)a_i - a\right)$. Because

$$\sum_{i=1}^{k}(k - i + 1)a_i = a\frac{k(k+1)(k+2)}{n(n+1)(n+2)}, \quad (20)$$

the sectional shape of the part with $\frac{n-k}{n+1}h \leq x \leq \frac{n-k+1}{n+1}h$ is

$$E_{k-1}E_k\colon y = \frac{1}{h}\left\{b - \frac{3ak(k+1)}{n(n+2)}\right\}x + a\frac{k(k+1)(3n - 2k + 2)}{n(n+1)(n+2)} - a. \quad (21)$$

If $D = E_{-1}$ and $O = E_n$, (21) holds for $0 \leq k \leq n$. Therefore, sectional shape $DE_0E_1 \cdots E_{n-1}O$ can be determined by (21).

2.2.2. Derivation of Warping Curve Based on Design Method of SS

Here, by finding the envelope of sectional shape $DE_0E_1 \cdots E_{n-1}O$, we obtain a warping curve that approximates the sectional shape between D and O when $n \to \infty$. (21) can be transformed as

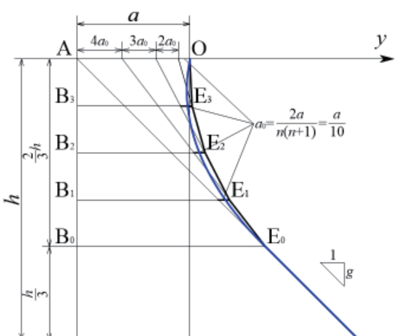


Figure 2. Example of warping curve (blue) based on GM that overhangs but does not hang over sectional shape when $n = 4$

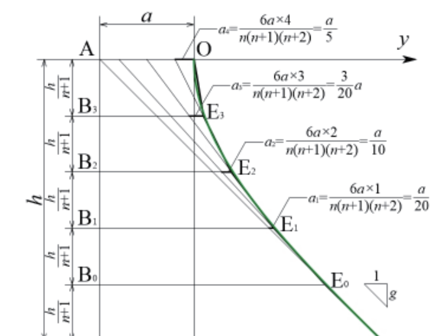


Figure 3. Design method for sectional shape based on *Sekisho Sho* (SS, for $n = 4$) and warping curve for $n \to \infty$ (green)

$$y=\frac{1}{h}\left\{b-\frac{3a\frac{k}{n}\left(\frac{k}{n}+\frac{1}{n}\right)}{1+\frac{2}{n}}\right\}x+a\frac{\frac{k}{n}\left(\frac{k}{n}+\frac{1}{n}\right)\left(3-2\frac{k}{n}+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}-a. \quad (21')$$

If we replace $\frac{k}{n}$ with variable t and let $n\to\infty$, we have

$$y=\frac{1}{h}(b-3at^2)x-at^2(2t-3)-a. \quad (22)$$

Multiplying both sides by h and transposing so that the right side is zero, we have

$$hy-(b-3at^2)x+aht^2(2t-3)+ah=0. \quad (23)$$

Partial differentiation of both sides by t yields

$$6atx+ah(6t^2-6t)=0. \quad (24)$$

Because $t=\frac{k}{n}$ is not identically zero,

$$t=1-\frac{x}{h}. \quad (25)$$

Substituting this into (22), we have

$$y=-a\left(\frac{x}{h}-1\right)^3+\frac{b}{h}x-a. \quad (26)$$

This envelope passes through two points, D and O. From the above, the sectional shape between D and O can be approximated by a quadratic function (Figure 3). Nishida et al. (2003) derived this cubic function almost the same as this one with the origin at point A. It is the same mathematical formula as the one in (26) with the $-a$ term removed[3], and it is the same curve. Let this be the warping curve between D and O.

2.2.3. Redefinition of Mathematical Formula for Warping Curve of SS

If we rewrite (26) using depth $d=b-a$ and the gradient of line DE_0 $g=\frac{h}{b}$ (initial gradient) instead of a and b, we have

$$y=\left(d-\frac{h}{g}\right)\left(\frac{x}{h}-1\right)^3+\frac{1}{g}x+d-\frac{h}{g}. \quad (27)$$

The initial gradient of the *ishigaki* is defined as the gradient of one of the $n+1$ from the bottom of the wall. Therefore, the larger n is, the more difficult it becomes to measure accurately on the actual *ishigaki*.

There is also a problem in treating this initial gradient as equivalent to the initial gradient of the GM, which is the lower third gradient of the *ishigaki*. Therefore, instead of the initial gradient g, we use the average gradient g_t of the lower third of the *ishigaki* (hereinafter referred to as the lower third gradient), which corresponds to the initial gradient of the GM, to represent the warping curve. Specifically, as shown in Figure 4, let E_t be the intersection point between the warping curve and line $x=\frac{2}{3}h$, and let g_t be the reciprocal of the slope of line DE_t. According to (26) and (27), because the y-coordinate of Et is $\frac{2}{3}\left(b-\frac{13}{9}a\right)=\frac{2}{27}\left(13d-\frac{4h}{g}\right)$,

$$g_t=\frac{9h}{9b-a}=\frac{9}{\frac{d}{h}+\frac{8}{g}}. \quad (28)$$

Substituting

$$\frac{1}{g}=\frac{1}{8}\left(\frac{9}{g_t}-\frac{d}{h}\right) \quad (29)$$

into (27), we have

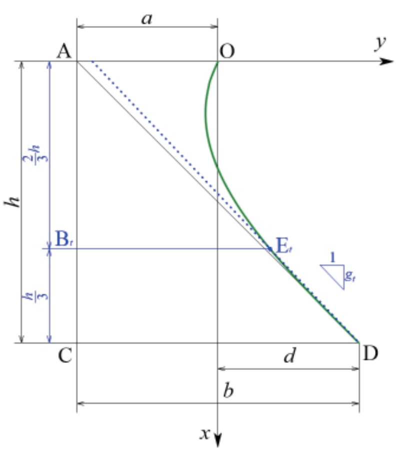


Figure 4. Example of overhang of warping curve (green) based on SS and definition of lower third gradient g_t

$$y=\frac{9}{8}\left(d-\frac{h}{g_t}\right)\left(\frac{x}{h}-1\right)^3-\frac{1}{8}\left(\frac{d}{h}-\frac{9}{g_t}\right)x+\frac{9}{8}\left(d-\frac{h}{g_t}\right). \quad (30)$$

2.2.4. Condition for No Overhang for Warping Curve of SS

Differentiating both sides of equation (26) by x, we have

$$\frac{dy}{dx}=-\frac{3a}{h}\left(\frac{x}{h}-1\right)^2+\frac{b}{h}. \quad (31)$$

Because $x\le h$,

$$\frac{d^2y}{dx^2}=-\frac{6a}{h^2}\left(\frac{x}{h}-1\right)\ge 0. \quad (32)$$

Therefore, $\frac{dy}{dx}$ is monotonically increasing for $0\le x\le h$. Because

$$\left.\frac{dy}{dx}\right|_{x=h}=\frac{b}{h}>0, \quad (33)$$

the condition that $y\ge 0$ for $0\le x\le h$ is

$$\left.\frac{dy}{dx}\right|_{x=0}=-\frac{3a}{h}+\frac{b}{h}\ge 0. \quad (34)$$

Therefore, we have

$$b\ge 3a. \quad (35)$$

This is the same as (12), indicating that the condition for the top of the *ishigaki* to not overhang is the same as in the GM. If we rewrite the condition using the initial gradient g instead of the upper and lower bottoms a and b, we also get

$$\frac{2h}{3d}\le g<\frac{h}{d} \quad (36)$$

as in the GM. Further rewriting the condition in (36) using the lower third gradient g_t, we have

$$\frac{9h}{13d}\le g_t<\frac{h}{d} \quad (37)$$

according to (29). Because $\frac{2h}{3d}<\frac{9h}{13d}$, the condition for no overhang is more severe for the SS than for the GM, where $g=g_t$, when compared with the lower third gradient. The lower third gradient of SS must be greater than or equal to 9/13 of the average gradient of the entire *ishigaki* to avoid overhang.

2.2.5. Practical Condition for No Overhang for Ishigaki by SS

In practical use, n is a finite natural number. Therefore, there is no problem if section $E_{n-1}O$ of the uppermost *ishigaki* does not overhang, i.e., the slope of line $E_{n-1}E_n$ is greater than or equal to zero. According to (21),

$$\frac{1}{h}\left\{b-\frac{3a(n+1)}{n+2}\right\}\ge 0. \quad (38)$$

Therefore, to divide line segment AB0 into n equal parts, i.e., to divide line segment AC into $n+1$ equal parts, it is sufficient to satisfy the condition

$$b\ge 3a\left(1-\frac{1}{n+2}\right). \quad (39)$$

If we rewrite this condition with d, h, and g, we have

$$\frac{h}{3d}\left(2-\frac{1}{n+1}\right)\le g<\frac{h}{d}. \quad (40)$$

If $n\to\infty$, (39) and (40) coincide with (35) and (36), respectively.

We also consider rewriting the condition in (40) using the lower third gradient. g_t is the lower third gradient of the curve when $n\to\infty$. Because n in (40) is a finite natural number, we find the lower third gradient g_{tn} when n is a finite natural number. Specifically, as shown in Figure 5, let E_{tn} be the intersection point between sectional shape $DE_0E_1\cdots E_{n-1}O$ and line $x=\frac{2}{3}h$, and let g_{tn} be the reciprocal of the slope of line DE_{tn}. According to (21), the y-coordinate of E_{tn} can be expressed as $\frac{2}{3}b+a\frac{k(k+1)(n-2k)}{n(n+1)(n+2)}-a$, and we have

$$g_{tn} = \frac{h}{b - 3a\dfrac{k(k+1)(n-2k)}{n(n+1)(n+2)}}. \qquad (41)$$

Let l be a natural number.

(i) If $n = 3l - 2$,

because E_{tn} lies on line segment $E_{l-2}E_{l-1}$ (but with $E_{-1} = D$), $k = l - 1 = \frac{n-1}{3}$, and

$$g_{tn} = \frac{9n(n+1)}{\dfrac{2(2n+1)^2}{g} + \dfrac{d}{h}(n-1)(n+2)}. \qquad (42)$$

Substituting

$$\frac{1}{g} = \frac{1}{2(2n+1)^2}\left\{\frac{9}{g_{tn}}n(n+1) - \frac{d}{h}(n-1)(n+2)\right\} \qquad (43)$$

into (40), we have

$$\frac{h}{3d}\left(2 - \frac{1}{n+1}\right) \le \frac{2(2n+1)^2}{\dfrac{9}{g_{tn}}n(n+1) - \dfrac{d}{h}(n-1)(n+2)}. \qquad (44)$$

Therefore,

$$\frac{9n(n+1)h}{(13n^2 + 19n + 4)d} \le g_{tn} < \frac{h}{d} \quad (n = 1,4,7,\cdots). \qquad (45)$$

The range of g_{tn} obtained here agrees with (37) when $n \to \infty$.

(ii) If $n = 3l - 1$,

because $E_{tn} = E_{l-1}$, $k = l - 1 = \frac{n-2}{3}$, and

$$g_{tn} = \frac{9n(n+2)}{\dfrac{8(n+1)^2}{g} + \dfrac{d}{h}(n-2)(n+4)}. \qquad (46)$$

Substituting

$$\frac{1}{g} = \frac{1}{8(n+1)^2}\left\{\frac{9}{g_{tn}}n(n+2) - \frac{d}{h}(n-2)(n+4)\right\} \qquad (47)$$

into (40), we have

$$\frac{h}{3d}\left(2 - \frac{1}{n+1}\right) \le \frac{8(n+1)^2}{\dfrac{9}{g_{tn}}n(n+2) - \dfrac{d}{h}(n-2)(n+4)}. \qquad (48)$$

Therefore,

$$\frac{9n(2n+1)h}{(26n^2 + 25n + 8)d} \le g_{tn} < \frac{h}{d} \quad (n = 2,5,8,\cdots). \qquad (49)$$

The range of g_{tn} obtained here agrees with (37) when $n \to \infty$.

(iii) If $n = 3l$,

because E_{tn} lies on line segment $E_{l-1}E_l$, $k = l = \frac{n}{3}$, and

$$g_{tn} = \frac{9(n+1)(n+2)}{\dfrac{2(2n+3)^2}{g} + \dfrac{d}{h}n(n+3)}. \qquad (50)$$

Substituting

$$\frac{1}{g} = \frac{1}{2(2n+3)^2}\left\{\frac{9}{g_{tn}}(n+1)(n+2) - \frac{d}{h}n(n+3)\right\} \qquad (51)$$

into (40), we have

$$\frac{h}{3d}\left(2 - \frac{1}{n+1}\right) \le \frac{2(2n+3)^2}{\dfrac{9}{g_{tn}}(n+1)(n+2) - \dfrac{d}{h}n(n+3)}. \qquad (52)$$

Therefore,

$$\frac{9(n+1)(2n+1)h}{(26n^2 + 51n + 27)d} \le g_{tn} < \frac{h}{d} \quad (n = 3,6,9,\cdots). \qquad (53)$$

The range of g_{tn} obtained here agrees with (37) when $n \to \infty$.

2.3. *ISHIGAKI HIDEN-NO-SHO* (IH)

The *Ishigaki Hiden-no-sho* (IH), which means "*Ishigaki* Secret Document," was already known in the *Kitagawa-bon*, written in 1743 by Sakubei Kitagawa, *ano* of the Kumamoto-han (Kumamoto Domain, based at Kumamoto Castle; Kyushu Research Institute for Cultural Properties, 2001). In recent years, the *Noguchi-bon*, written in 1680, was also reprinted by the Ishikawa Prefectural Research Institute of Kanazawa Castle (2011). The contents of the two versions are almost identical.

2.3.1. Design Method by IH

According to Kitagaki (1987) and Nishida et al. (2003), the design method described in IH can be formulated as follows from (a) to (f). In Figure 6, as in Figures 1 and 3, the upper end of the curve is the origin O, the x-axis is downward vertical, and the y-axis is outward horizontal (in front of the *ishigaki*).

(a) Determine the height $h = \mathrm{AC}$, the upper bottom $a = \mathrm{OA}$, and the lower bottom $b = \mathrm{CD}$.

(b) Divide line segment AC into equal parts by natural number $n+1$, and let $B_0, B_1, \ldots, B_{n-1}$ be the points closest to C in order.

(c) The sectional shape of part n+1 from the bottom, i.e., $\frac{n}{n+1}h \le x \le h$, is line DA. Let this sectional shape be line segment DE_0, which is $E_0\left(\frac{n}{n+1}h, \frac{n}{n+1}b - a\right)$, and

$$DE_0\colon\ y = \frac{b}{h}x - a. \qquad (54)$$

(d) At $i = 1,2,\cdots,n$, we define a_i by

$$a_i = \frac{a}{n(n-i+1)}. \qquad (55)$$

(e) The point that protrudes a_1 from the intersection of the straight line drawn horizontally from B_1 in the direction of the y-axis and straight line DE_0 to the front of the *ishigaki* is $E_1\left(\frac{n-1}{n+1}h, \frac{n-1}{n+1}b + a_1 - a\right)$. The sectional shape of the part with $\frac{n-1}{n+1}h \le x \le \frac{n}{n+1}h$ is

$$E_0E_1\colon y = \frac{1}{h}\left\{b - \frac{a}{n}\left(1 + \frac{1}{n}\right)\right\}x + \frac{a}{n} - a. \qquad (56)$$

(f) Similarly, when $2 \le k \le n-1$, the point that protrudes a_k from the intersection of the straight line drawn horizontally from B_k in the direction of the y-axis and straight line $E_{k-2}E_{k-1}$ to the front of the *ishigaki* can be obtained inductively as $E_k\left(\frac{n-k}{n+1}h, \frac{n-k}{n+1}b + \sum_{i=1}^{k}(k-i+1)a_i - a\right)$. The sectional shape of the part with $\frac{n-k}{n+1}h \le x \le \frac{n-k+1}{n+1}h$ is

$$E_{k-1}E_k\colon y = \frac{1}{h}\left\{b - a\left(1 + \frac{1}{n}\right)\sum_{i=1}^{k}\frac{1}{n-i+1}\right\}x + a\frac{k}{n} - a. \quad (57)$$

If $D = E_{-1}$ and $O = E_n$, and we define $\sum_{i=1}^{0}\frac{1}{n-i+1} = 0$, then (57) is valid for $0 \le k \le n$. Therefore, sectional shape

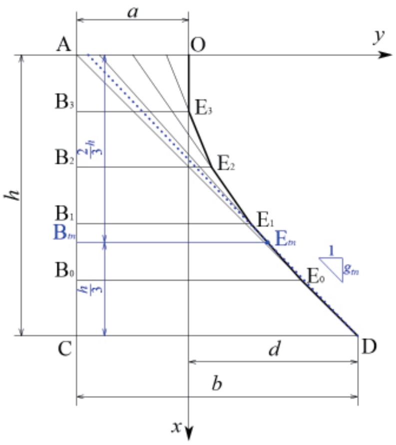


Figure 5. Definition of lower third gradient g_{tn} for case $n = 4$ in sectional shape based on SS

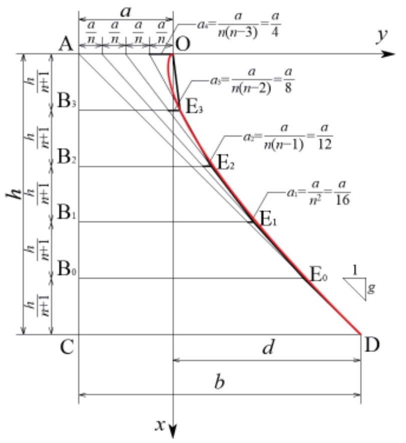


Figure 6. Design method for sectional shape based on *Ishigaki Hiden-no-sho* (IH, for $n = 4$) and warping curve for $n \to \infty$ (red)

$\mathrm{DE_0E_1 \cdots E_{n-1}O}$ can be determined by (57).

2.3.2. Derivation of Warping Curve Based on Design Method of IH

Here, by finding the envelope of sectional shape $\mathrm{DE_0E_1 \cdots E_{n-1}O}$, we obtain a warping curve that approximates the sectional shape between D and O when $n \to \infty$. If $f(x)$ is a Riemann integrable,

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f\left(a+(b-a)\frac{j}{n}\right) = \int_a^b f(x)dx. \quad (58)$$

If $a = 1, b = n+1, f(x) = \frac{1}{x}$, we have

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j+1} = \lim_{n\to\infty} \log(n+1). \quad (59)$$

If we substitute n for $n-k$ in this equation, we have

$$\lim_{n\to\infty} \sum_{j=1}^{n-k} \frac{1}{j+1} = \lim_{n\to\infty} \log(n-k+1). \quad (60)$$

Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{k} \frac{1}{n-i+1} \\ &= \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j+1} - \frac{1}{n+1} - \sum_{j=1}^{n-k} \frac{1}{j+1} + \frac{1}{n-k+1} \right) \\ &= \lim_{n\to\infty} \{\log(n+1) - \log(n-k+1)\} \\ &= \lim_{n\to\infty} \log\left(\frac{1+\frac{1}{n}}{1-\frac{k}{n}+\frac{1}{n}} \right). \end{aligned} \quad (61)$$

Now, if we set $\frac{k}{n} = t$ to find the envelope,

$$\lim_{n\to\infty} \sum_{i=1}^{k} \frac{1}{n-i+1} = \log\left(\frac{1}{1-t}\right) = -\log(1-t). \quad (62)$$

Therefore, in (57), if we replace $\frac{k}{n}$ with variable t and let $n \to \infty$, we have

$$y = \frac{1}{h}\{b + a\log(1-t)\}x + at - a. \quad (63)$$

Multiplying both sides by h and transposing so that the right side is zero, we have

$$hy - \{b + a\log(1-t)\}x - aht + ah = 0. \quad (64)$$

Partial differentiation of both sides by t yields

$$\frac{a}{1-t}x - ah = 0. \quad (65)$$

If we find t,

$$t = 1 - \frac{x}{h}. \quad (66)$$

Substituting this into (63), we have

$$y = \frac{1}{h}(b-a)x + \frac{a}{h}x\log\left(\frac{x}{h}\right). \quad (67)$$

This envelope passes through D. Also, (67) is defined in the range of $x > 0$, but according to L'Hôpital's rule,

$$\lim_{x\to+0} x\log\left(\frac{x}{h}\right) = \lim_{x\to+0} \frac{\log\left(\frac{x}{h}\right)}{\frac{1}{x}} = \lim_{x\to+0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0. \quad (68)$$

Therefore, when $x \to +0$, y approaches 0, i.e., the origin O. From the above, the curve defined by (67) is a curve that approximates the sectional shape between D and O (Figure 6).

Yanai (1988) derived this function with the origin at point A. It is the same as the right side of (67) with the addition of term a, and it is the same curve. The same is true for the function shown by Nishida et al. (2003)[4]. Let this be the warping curve between D and O.

2.3.3. Redefinition of Mathematical Formula for Warping Curve of IH

If we rewrite (67) using the depth $d = b - a$ and the gradient of line $\mathrm{DE_0}$ $g = \frac{h}{b}$ (initial gradient) instead of a and b, we have

$$y = \frac{d}{h}x + \left(\frac{1}{g} - \frac{d}{h}\right)x\log\left(\frac{x}{h}\right). \quad (69)$$

In the same way as in the SS, let $\mathrm{E_t}$ be the intersection point between the warping curve and line $x = \frac{2}{3}h$, and let g_t be the reciprocal of the slope of line $\mathrm{DE_t}$ to get the lower third gradient g_t, which corresponds to the initial gradient of the GM. According to (67) and (69), because the y-coordinate of Et is $\frac{2}{3}(b-a) - \frac{2}{3}a\log\left(\frac{3}{2}\right) = \frac{2}{3}d - \frac{2}{3}\left(\frac{h}{g} - d\right)\log\left(\frac{3}{2}\right)$,

$$g_t = \frac{h}{b - a + 2a\log\left(\frac{3}{2}\right)} = \frac{h}{d + 2\left(\frac{h}{g} - d\right)\log\left(\frac{3}{2}\right)}. \quad (70)$$

Substituting

$$\frac{1}{g} = \frac{d}{h} + \frac{\frac{1}{g_t} - \frac{d}{h}}{2\log\left(\frac{3}{2}\right)} \quad (71)$$

into (69), we have

$$y = \frac{d}{h}x + \frac{\frac{1}{g_t} - \frac{d}{h}}{2\log\left(\frac{3}{2}\right)}x\log\left(\frac{x}{h}\right). \quad (72)$$

2.3.4. Practical Condition for No Overhang for Ishigaki by IH

Differentiating both sides of (67) by x,

$$\frac{dy}{dx} = \frac{b}{h} + \frac{a}{h}\log\left(\frac{x}{h}\right). \quad (73)$$

Therefore,

$$\lim_{x\to+0} \frac{dy}{dx} = -\infty. \quad (74)$$

Because the slope of the curve based on the IH is negative near the origin O, as Fujii (2018) already pointed out, the upper part of the curve always overhangs, at least locally. However, for practical use, n is a finite natural number. Therefore, if section $\mathrm{E_{n-1}O}$ of the uppermost *ishigaki* does not overhang, i.e., the slope of line $\mathrm{E_{n-1}E_n}$ is greater than or equal to zero, there is no problem. According to (57),

$$\frac{1}{h}\left\{b - a\left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{n-i+1}\right\} \geq 0. \quad (75)$$

Therefore, to divide line segment $\mathrm{AB_0}$ into n equal parts, i.e., to divide line segment AC into $n+1$ equal parts, it is sufficient to satisfy the condition of

$$b \geq a\left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{i}. \quad (76)$$

Also, if we rewrite the condition using the initial gradient g instead of the upper and lower bottoms a and b, because a is not positive when $g \geq \frac{h}{d}$,

$$\frac{h}{d}\left\{1 - \frac{1}{\left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{i}}\right\} \leq g < \frac{h}{d}. \quad (77)$$

We also consider rewriting the condition in (77) using the lower third gradient. We find the lower third gradient g_{tn} when n is a finite natural number, as in the SS. Specifically, let E_{tn} be the intersection point between sectional shape $\mathrm{DE_0E_1 \cdots E_{n-1}O}$ and line $x = \frac{2}{3}h$, and let g_{tn} be the reciprocal of the slope of line DE_{tn}. According to (57), the y-coordinate of E_{tn} can be expressed as $\frac{2}{3}\left\{b - a\left(1+\frac{1}{n}\right)\sum_{i=1}^{k}\frac{1}{n-i+1}\right\} + a\frac{k}{n} - a$, and we have

$$g_{tn} = \frac{h}{b + a\left(1+\frac{1}{n}\right)\sum_{i=1}^{k}\frac{1}{n-i+1} - 3a\frac{k}{n}}. \tag{78}$$

Let l be a natural number.

(i) If $n = 3l - 2$,

because E_{tn} lies on line segment $E_{l-2}E_{l-1}$ (but with $E_{-1} = D$), $k = l - 1 = \frac{n-1}{3}$. For $k \geq 1$, that is, $n \geq 4$,

$$g_{tn} = \frac{1}{\frac{1}{g} + \left(\frac{1}{g} - \frac{d}{h}\right)\left\{2\left(1+\frac{1}{n}\right)\sum_{i=1}^{\frac{n-1}{3}}\frac{1}{n-i+1} - 1 + \frac{1}{n}\right\}}. \tag{79}$$

Substituting

$$\frac{1}{g} = \frac{1}{\frac{1}{n} + 2\left(1+\frac{1}{n}\right)\sum_{i=\frac{2(n+2)}{3}}^{n}\frac{1}{i}}\left(\frac{1}{g_{tn}} - \frac{d}{h}\right) + \frac{d}{h} \tag{80}$$

into (77), we have

$$\frac{h}{d}\left\{1 - \frac{1}{\left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{i}}\right\} \leq \frac{1}{\frac{1}{\frac{1}{n} + 2\left(1+\frac{1}{n}\right)\sum_{i=\frac{2(n+2)}{3}}^{n}\frac{1}{i}}\left(\frac{1}{g_{tn}} - \frac{d}{h}\right) + \frac{d}{h}}. \tag{81}$$

Therefore,

$$\frac{h}{d}\left(\frac{\sum_{i=1}^{n}\frac{1}{i} - 1 + \frac{1}{n+1}}{3\sum_{i=1}^{n}\frac{1}{i} - 2\sum_{i=1}^{\frac{2n+1}{3}}\frac{1}{i} - 1 + \frac{2}{n+1}}\right) \leq g_{tn} < \frac{h}{d}$$

$$(n = 4, 7, 10, \cdots). \tag{82}$$

If $k = 0$, that is, $n = 1$, because the y-coordinate of E_{t1} is $\frac{2}{3}b - a$,

$$g_{t1} = \frac{h}{b} = g. \tag{83}$$

According to (77),

$$\frac{h}{2d} \leq g_{t1} < \frac{h}{d}. \tag{84}$$

This (84) is equal to (82) with $n = 1$ substituted.

(ii) If $n = 3l - 1$,

because $E_{tn} = E_{l-1}$, $k = l - 1 = \frac{n-2}{3}$. For $k \geq 1$, that is, $n \geq 5$,

$$g_{tn} = \frac{1}{\frac{1}{g} + \left(\frac{1}{g} - \frac{d}{h}\right)\left\{2\left(1+\frac{1}{n}\right)\sum_{i=1}^{\frac{n-2}{3}}\frac{1}{n-i+1} - 1 + \frac{2}{n}\right\}}. \tag{85}$$

Substituting

$$\frac{1}{g} = \frac{1}{\frac{2}{n} + 2\left(1+\frac{1}{n}\right)\sum_{i=\frac{2n+5}{3}}^{n}\frac{1}{i}}\left(\frac{1}{g_{tn}} - \frac{d}{h}\right) + \frac{d}{h} \tag{86}$$

into (77), we have

$$\frac{h}{d}\left\{1 - \frac{1}{\left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{i}}\right\} \leq \frac{1}{\frac{1}{\frac{2}{n} + 2\left(1+\frac{1}{n}\right)\sum_{i=\frac{2n+5}{3}}^{n}\frac{1}{i}}\left(\frac{1}{g_{tn}} - \frac{d}{h}\right) + \frac{d}{h}}. \tag{87}$$

Therefore,

$$\frac{h}{d}\left(\frac{\sum_{i=1}^{n}\frac{1}{i} - 1 + \frac{1}{n+1}}{3\sum_{i=1}^{n}\frac{1}{i} - 2\sum_{i=1}^{\frac{2(n+1)}{3}}\frac{1}{i} - 1 + \frac{3}{n+1}}\right) \leq g_{tn} < \frac{h}{d}.$$

$$(n = 5, 8, 11, \cdots). \tag{88}$$

If $k = 0$, that is, $n = 2$, because the y-coordinate of E_{t2} is $\frac{2}{3}b - a$,

$$g_{t2} = \frac{h}{b} = g. \tag{89}$$

According to (77),

$$\frac{5h}{9d} \leq g_{t2} < \frac{h}{d}. \tag{90}$$

This (90) is equal to (88) with $n = 2$ substituted.

(iii) If $n = 3l$,

because E_{tn} lies on line segment $E_{l-1}E_l$, $k = l = \frac{n}{3}$, and

$$g_{tn} = \frac{1}{\frac{1}{g} + \left(\frac{1}{g} - \frac{d}{h}\right)\left\{2\left(1+\frac{1}{n}\right)\sum_{i=1}^{\frac{n}{3}}\frac{1}{n-i+1} - 1\right\}}. \tag{91}$$

Substituting

$$\frac{1}{g} = \frac{1}{2\left(1+\frac{1}{n}\right)\sum_{i=\frac{2}{3}n+1}^{n}\frac{1}{i}}\left(\frac{1}{g_{tn}} - \frac{d}{h}\right) + \frac{d}{h} \tag{92}$$

into (77), we have

$$\frac{h}{d}\left\{1 - \frac{1}{\left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{i}}\right\} \leq \frac{1}{\frac{1}{2\left(1+\frac{1}{n}\right)\sum_{i=\frac{2}{3}n+1}^{n}\frac{1}{i}}\left(\frac{1}{g_{tn}} - \frac{d}{h}\right) + \frac{d}{h}}. \tag{93}$$

Therefore,

$$\frac{h}{d}\left(\frac{\sum_{i=1}^{n}\frac{1}{i} - 1 + \frac{1}{n+1}}{3\sum_{i=1}^{n}\frac{1}{i} - 2\sum_{i=1}^{\frac{2}{3}n}\frac{1}{i} - 1 + \frac{1}{n+1}}\right) \leq g_{tn} < \frac{h}{d}$$

$$(n = 3, 6, 9, \cdots). \tag{94}$$

3. Comparison of Practical Conditions for No Overhang for *Ishigaki*

Figure 7 shows the results of comparing the conditions of the initial gradient where the top of the *ishigaki* does not overhang in practical use with the lower limit of $\frac{gd}{h}$ (the ratio of the initial gradient to the average gradient of the entire *ishigaki*) obtained by (16), (40), and (77).

One problem that arose during the comparison is that, for example, when designing an *ishigaki* by dividing its entire height into six equal parts, the *Goto-ke Monjo* (GM, Figure 1) divides its upper two-thirds into n equal parts, resulting in $n = 4$, while the *Sekisho Sho* (SS, Figure 3) and *Ishigaki Hiden-no-sho* (IH, Figure 6) divide the entire *ishigaki* into $n + 1$ equal parts, resulting in $n = 5$. As shown in this example, even if an *ishigaki* of the same height is divided into equal parts at the same interval, n differs depending on the design method. Therefore, the number of divisions (m), which indicates the number of divisions of the entire *ishigaki*, was taken as the horizontal axis for comparison. In the SM, $m = \frac{3}{2}n$ was used, and the height of the *ishigaki* was considered to be divided into $\frac{3}{2}n$ equal parts from the top. In the SS and IH, $m = n + 1$ was used.

The GM and SS both have a $\frac{gd}{h}$ of $\frac{2}{3}$ when $n \to \infty$, but the convergence is slower in the GM than in the SS. Therefore, if n is finite, the conditions of the GM are looser than those of the SS. On the other hand, the IH has the biggest $\frac{gd}{h}$ for all n, and the conditions are more severe.

In the same way, the results for the lower third gradient condition, where the top of the *ishigaki* does not overhang in practical use, are compared at the lower limit of $\frac{g_{tn}d}{h}$ (the ratio of the initial gradient to the average gradient of the entire *ishigaki*) in Figure 8. In the GM, $\frac{g_{tn}d}{h} = \frac{gd}{h}$ because the initial gradient and the lower third gradient are equal. The values of the SS are (45), (49), and (53), and the values of the IH are (82), (84), (88), (90), and (94). Comparing with Figure 7, it can also be seen that the conditions for the GM are the loosest, and those for the IH are the strictest, but the conditions for the SS and the IH are stricter than in Figure 7.

4. Comparison with Photogrammetric Results of Edges of *Ishigaki* at Hikone Castle

The three warping curves with the formula redefined for the lower third gradient were compared with the photogrammetric results obtained at three places near the edges of the *ishigaki* at Hikone Castle (Umezaki et al., 2021)[5]. The procedure was as follows from (i) to (iii).

(i) The point clouds of the photogrammetric results near the three edges were parallel projected at the two angles shown in Figure 9. In reality, people observe *ishigaki* from different angles. However, in this paper, we decided to project the *ishigaki* not from the front elevation but from the angle rotated by 30 degrees[6]. By doing so, the curve of the edge of the *ishigaki* not only becomes an occluding edge (Gibson, 1979), but unevenness in the *ishigaki* on the back side does not appear as well, making it easier to observe the edge.

(ii) We measured the depth d and the lower third gradient g_t from the height h shown in the previous report and the projection of the point cloud from the photogrammetry results (Table 1). According to the angle of the projection, d is $\frac{2}{\sqrt{3}}$ times the value measured on the projection plane, and g_t is $\frac{\sqrt{3}}{2}$ times the value because the width of the projection of the *ishigaki* in Figures 10, 11, and 12 is $\frac{\sqrt{3}}{2}$ times its elevation[7].

(iii) Substituting h and the measured d and g_t into (10), (30), and (72), the three kinds of warping curves of each *ishigaki* and the projection were drawn and compared with the point clouds of the photogrammetric results [$g = g_t$ in (10) for GM]. The results of substituting h, d, and g_t of TS-L, NS-L, and TY-L in Table 1 into the equations and comparing them with the point clouds of the photogrammetric results are shown in Figures 10, 11, and 12.

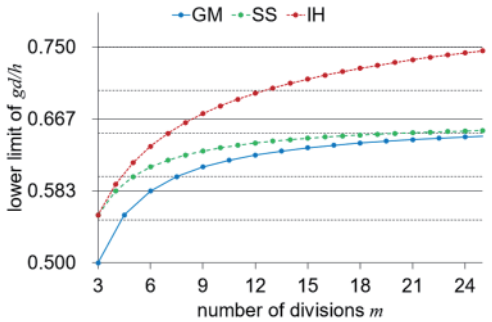


Figure 7. Comparison of initial gradient conditions where *ishigaki* does not overhang for practical use

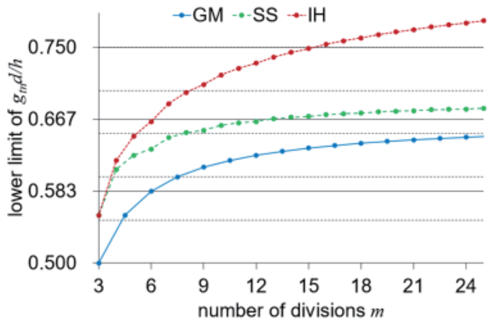
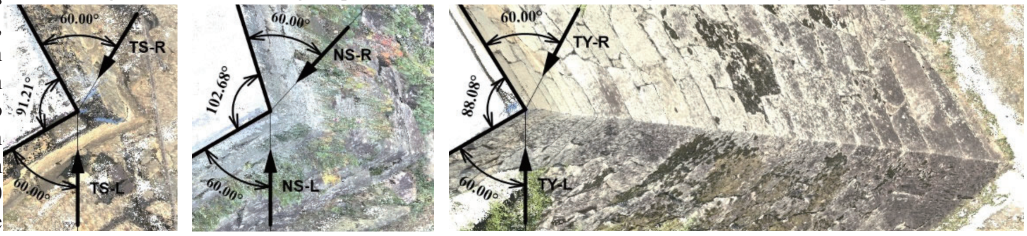


Figure 8. Comparison of lower third gradient conditions where *ishigaki* does not overhang for practical use



Tenshu *Nishinomaru Sanju Yagura* *Tenbin Yagura*

Figure 9. Plan for each photogrammetric result for each *ishigaki* and angles of projection S=1:100

Table 1. Measurement results for height h, depth d, and lower third gradient g_t of each *ishigaki* and projection

ishigaki-projection	h[mm]	d[mm]	g_t
TS-L	4,239	964	2.753
TS-R		955	2.737
NS-L	5,864	2,400	1.902
NS-R		2,166	2.137
TY-L	10,680	6,960	1.292
TY-R		6,508	1.379

4.1. COMPARISON WITH *ISHIGAKI* OF *TENSHU* (Figure 10)

This *ishigaki* was built between 1604 and 1607 during the construction of Hikone Castle. Most of the stones are roughly split and processed, and there are few hewn stones. It is considered to be one of the earliest *ishigaki* at Hikone Castle (Hikone Educational Bureau, 2010). The photogrammetry was done near the southwest edge.

Because the value of $\frac{g_t d}{h}$ is 0.626 for TS-L and 0.617 for TS-R, which are smaller than $\frac{2}{3} \simeq 0.667$, all three types of warping curves had an overhang. Moreover, because the stones of the edge are stacked in eight tiers, if we assume that the height of the *ishigaki* is divided into eight equal parts and $n = 7$ in (45) and (82), the lower limit of Figure 8 is 0.651 in the SS and 0.698 in the IH. Therefore, the *ishigaki* designed with the SS and IH are considered to overhang even in practical use. The GM does not correspond to the eighth division of the height of the *ishigaki*, but if we assume that it is divided into nine equal parts and $n = 6$ in (16), the lower limit is 0.611, which is smaller than 0.617. Therefore, if we design this *ishigaki* with the GM, it will not have an overhang in practical use.

Due to the low quality of the stone processing, it is difficult to find a curve that is close, but the curve of the edge of this *ishigaki* is relatively closer to the curves in the GM and the SS than to the IH. The distance between the curves in the GM and the SS is less than 10 mm.

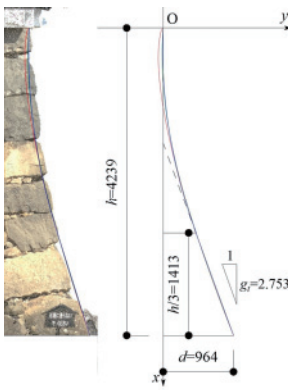


Figure 10. Curves of TS-L (right) and comparison with projection of photogrammetric result (left) S=1:100 (blue: GM, green: SS, red: IH). Width of projection (left) is $\frac{\sqrt{3}}{2} \simeq 0.866$ times that of actual *ishigaki*.

4.2. COMPARISON WITH *ISHIGAKI* OF *NISHINOMARU SANJU YAGURA* (Figure 11)

This *ishigaki* was repaired in 1852. The stones are hewn stone, which is highly processed, but old materials were used. Therefore, the size and shape of the stones are not very consistent (Hikone Educational Bureau, 2010). The photogrammetry was done near the northeast edge.

Because the value of $\frac{g_t d}{h}$ is 0.778 for NS-L and 0.789 for NS-R, which is larger than $\frac{9}{13} \simeq 0.692$, the warping curves based on the GM and the SS do not overhang. Moreover, because the stones of the edge are stacked in 13 tiers, if we assume that the height of the *ishigaki* is divided into 13 equal parts and $n = 12$ in (94) of the IH, the lower limit of Figure 8 is 0.739. Because $\frac{g_t d}{h}$ is larger than this value, the *ishigaki* designed by the SS will also not have an overhang in practical use. Therefore, we can say that no matter which design method is used to design this *ishigaki*, it will not have an overhang in practical use.

The curve of the edge of this *ishigaki* is relatively closer to the curves in the GM and the SS than to the IH. The distance between the curves in the GM and the SS is less than 11 mm.

4.3. COMPARISON WITH *ISHIGAKI* OF *TENBIN YAGURA* (Figure 12)

This *ishigaki* was re-stacked around 1854. Like the *Nishinomaru*

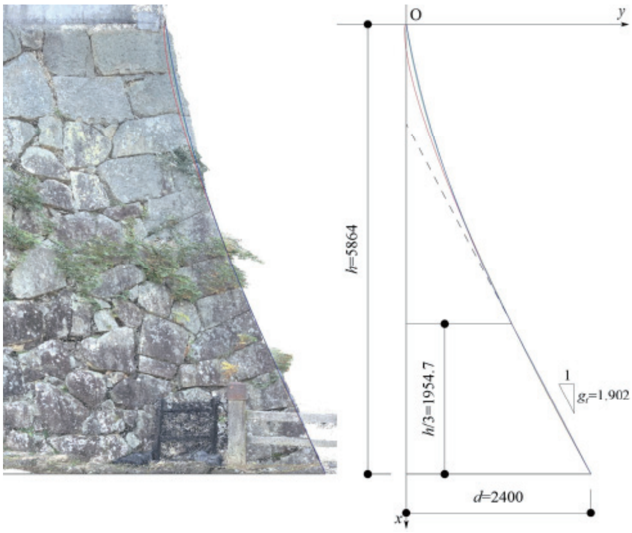


Figure 11. Curves of NS-L (right) and comparison with projection of photogrammetric result (left) S=1:100 (blue: GM, green: SS, red: IH). Width of projection (left) is $\frac{\sqrt{3}}{2} \simeq 0.866$ times that of actual *ishigaki*.

Sanju Yagura, hewn stones are used, but the stones are processed into almost rectangular shapes, and the size and shape are relatively consistent. This is the most recent of the *ishigaki* at Hikone Castle from the Edo period (Hikone Educational Bureau, 2010). The photogrammetry was done near the northwest edge.

The value of $\frac{g_t d}{h}$ is 0.841 for TY-L and 0.840 for TY-R, which is larger than $\frac{9}{13} \simeq 0.692$. Therefore, the warping curves based on the GM and the SS do not overhang. Moreover, because the stones of the edge are stacked in 25 tiers, if we assume that the height of the *ishigaki* is divided into 25 equal parts and $n = 24$ in (94) of the IH, the lower limit of Figure 8 is 0.781. Because $\frac{g_t d}{h}$ is larger than this value, the *ishigaki* designed with the SS will also not overhang in practical use. Therefore, we can say that no matter which design method is used to design this *ishigaki*, it will not overhang in practical use.

The curve of the edge of the *ishigaki* is close to the curve of the IH except for the top four tiers. At the top, the curve is close to that of the GM and SS. The distance between the curves in the GM and the SS is less than 21 mm.

5. Conclusion

In this paper, we redefined the mathematical formulae for warping curves obtained from three documents describing the design methods of warping curves of *ishigaki* (stone walls) at Japanese castles, *Goto-ke Monjo* (GM), *Sekisho Sho* (SS), and *Ishigaki Hiden-no-sho* (IH), as mathematical formulae with parameters of depth, height, and initial gradient. We also proposed redefinitions of the mathematical formulae for the warping curves by selecting a lower third gradient (the average gradient of the lower third of the *ishigaki*) that is suitable for comparing differences in warping curves among the three design methods and existing *ishigaki* at castles. Furthermore, the conditions under which a warping curve based on each design method and an *ishigaki* designed on the basis of the methods do not overhang were compared. Finally, we compared the three redefined warping curves and the results of photogrammetry previously conducted near the edges of the *ishigaki* at Hikone Castle. The results of the study revealed the following.

1) In order for a warping curve based on the GM to not overhang, the initial gradient (equal to the lower third gradient) must be greater than

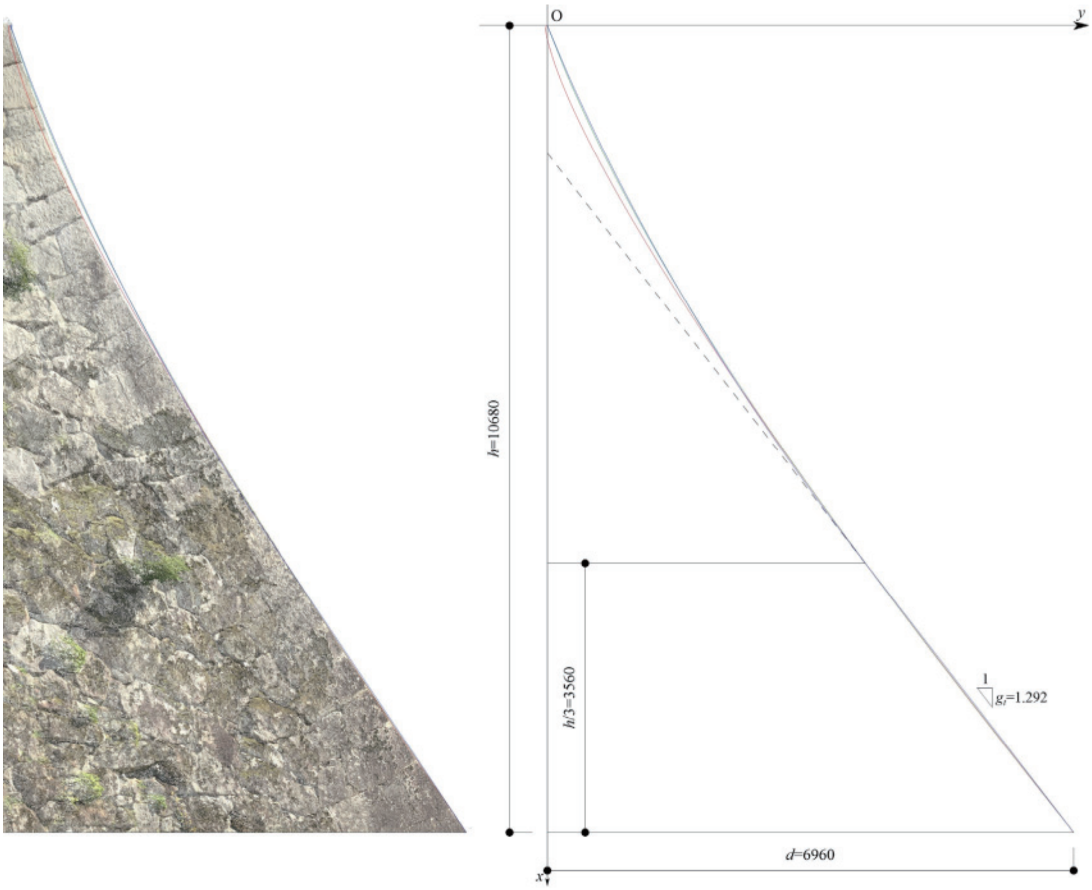


Figure 12. Curves of TY-L (right) and comparison with projection of photogrammetric result (left) S=1:100 (blue: GM, green: SS, red: IH). Width of projection (left) is $\frac{\sqrt{3}}{2} \simeq 0.866$ times that of actual *ishigaki*.

or equal to the average gradient of the entire *ishigaki*.
2) In order for a warping curve based on the SS to not overhang, the initial gradient must be greater than or equal to 2/3, or the lower third of the gradient must be greater than or equal to 9/13 of the average gradient of the entire *ishigaki*.
3) Comparing the conditions under which the tops of an *ishigaki* designed on the basis of each design method do not overhang, the conditions of the GM are the loosest, and those designed with the IH are the severest. The conditions in the SS and IH are more severe when compared with the same lower third gradient than when compared with the same initial gradient.
4) Substituting the depth, height, and lower third gradient of the three edges of the *ishigaki* at Hikone Castle into the mathematical formulae for the warping curves in the GM and the SS, we found that the two curves are close together, with only 10-21 mm separating them at most.
5) Under the conditions of the *ishigaki* of the *Tenshu*, the warping curve overhangs in all the design methods. However, considering the number of tiers of the *ishigaki*, the curve does not overhang in practical use only when it is designed on the basis of the GM.
6) Under the conditions of the *ishigaki* of the *Nishinomaru Sanju Yagura* and *Tenbin Yagura*, only the warping curves with the IH overhang, but considering the number of tiers of the *ishigaki*, none of the design methods cause overhangs in practical use.
7) The warping curves of the *ishigaki* of the *Tenshu* and the *Nishinomaru Sanju Yagura* are closer to the curves in the GM and the SS. The curve of the *ishigaki* of the *Tenbin Yagura*, which is the most recent *ishigaki* at Hikone Castle, is closer to the curve of the IH except for the uppermost four tiers.

Acknowledgements

This work was supported by JSPS KAKENHI Grant Numbers JP20K12520, JP21K13799, and JST CREST Grant Number JPMJCR1911.

Endnotes

1. Nishida et al. (2003), in their English abstract, refer to *Goto-ke Monjo* as *Gotou-ke Monjyo* and *Sekisho Sho* as *Sekishou-sho*.
2. The design method presented in the commentary by Bin Kinai at the end of the reprinted *Goto-ke Monjo* (Japan Sea Culture Research Institute, Kanazawa University, 1976) is different from that of this paper. In later years, Kinai (1988) presented two types of warping curves based on his design method, which are completely different from (9). However, Kinai's design method divides the upper bottom equally (see Figure 6). This idea is not shown in the *Goto-ke Monjo* and cannot be considered correct, as already pointed out by Morimoto et al. (2000).
3. Nishida et al. (2003) used $h_2 = \frac{n}{n+1}h$ and set $y = a\left(1 - \frac{x}{h_2}\right)^3 + \frac{b}{h}x$. Because $\lim_{n\to\infty} h_2 = h$, the mathematical formula becomes the same as (26) if we add term $-a$.
4. Nishida et al. (2003) used $h_2 = \frac{n}{n+1}h$ and set $y = \frac{a}{h_2}\left(\log\frac{x}{h_2} - 1\right) + \frac{b}{h}x + a$. Because $\lim_{n\to\infty} h_2 = h$, the mathematical formula becomes the same as (67) if we delete term $+a$.
5. Photogrammetry was conducted using Agisoft Metashape Professional. The photographs used for the photogrammetry were 100 of the *Tenshu*, 78 of the *Nishinomaru Sanju Yagura*, and 72 of the *Tenbin Yagura*, all of which were taken in November 2020.
6. In the previous paper (Umezaki et al. 2021), the angle of the projection was determined for each *ishigaki*. However, the angle is unified for all *ishigaki* in this paper.
7. The height h is the same as previously reported (Umezaki et al. 2021), but the depth d and the lower third gradient g_t were newly measured. Therefore, the upper bottom a and lower bottom b calculated from these values and the value of $g = g_t$ do not agree with the values estimated by the method based on the GM in the previous report. a and b cannot be measured directly from the photogrammetric results, but d and g_t can be measured. Therefore, the precision of measurements is considered to be improved.

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