Continuous Column Effects of Coupled Shear-Flexural-Beam Models in terms of Static Stability

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Abstract: The continuous column (C.C.) effects, in which elastic columns continuous over structural height prevent story failure mechanism and mitigate the drift concentration in particular stories during earthquakes for steel, reinforced concrete and wooden frame structures. The coupled shear-flexural-beam models, which was proposed and investigated by the author, are simplified models consisting with a shear-beam and a flexural-beam that can consider the C.C. effects explicitly. In this paper, the stiffness matrices of the shear-beam, the flexural-beam and that of the coupled shear-flexural-beam model are derived and their characteristics are investigated in terms of the stability. The eigenvalue analyses are conducted for the 2- and 3-story coupled shear-flexural-beam models at the elastic stage and the assumed 1st-story mechanism is investigated quantitatively to evaluate the C.C. effects on static stability of entire structure.

1. Introduction

Elastic columns and multi-story walls that are continuous over the structural height mitigate the drift concentration in the particular stories, prevent the story failure mechanism, and increase the stability during the earthquakes. This is referred to as the continuous column (C.C.) effects. Using the coupled shear-flexural-beam models as shown in Figure 1, it was verified that each story drift angle becomes more uniform as the flexural stiffness of the flexural-beam increases for steel, reinforced concrete frame structures subjected to earthquake motions. In this paper, the stiffness matrices of the shear-beam, flexural-beam and the coupled shear-flexural-beam model are derived and their characteristics are investigated in terms of the stability. Eigenvalue analyses are conducted for 2- and 3-story coupled shear-flexural-beam models to evaluate the C.C. effects on static stability. This C.C. effects may be related to a mystery of Japanese wooden five-story pagodas with a Shinbashira, penetrating a center of the tower, which have not been collapsed until now even subjected to massive earthquakes.

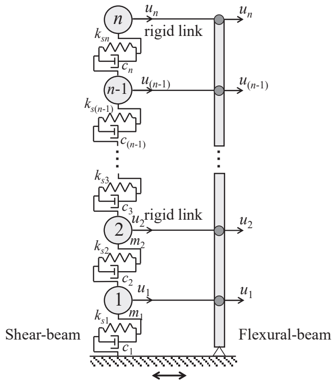


Figure 1. Coupled shear-flexural-beam model

2. Stiffness matrix of coupled shear-flexural-beam models

2.1. SHEAR-BEAM MODEL

Stiffness matrices of 2-, 3- and n-story shear-beam models as shown in Figure 2 are derived and these are investigated in terms of static stability.

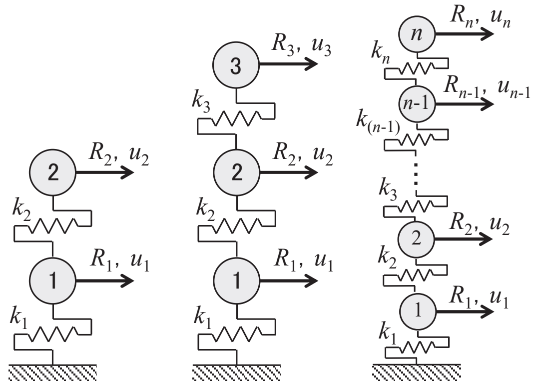


Figure 2. Shear-beam models

We will start with a 2-story shear-beam model as shown in Figure 2(a). Here, k_1 and k_2 is the stiffness of the 1st- and 2nd-story horizontal springs. The masses of the 1st- and 2nd-story are subjected to horizontal forces, R_1 and R_2, resulting in the horizontal displacements, u_1 and u_2, respectively. The relations of the horizontal forces and the displacements are given by (1). Therefore, the stiffness matrix of a 2-story shear-beam model, **K**, is given by (2).

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \mathbf{R}=\mathbf{K}\mathbf{u} \tag{1}$$

$$\mathbf{K}_{2\times 2} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{2}$$

The determinant of the stiffness matrix **K** is calculated as (3).

$$\det \mathbf{K}_{2\times 2} = (k_1+k_2)k_2 - (-k_2)(-k_2) = k_1k_2 \tag{3}$$

The eigenvalues of **K** can be calculated as the solutions of the characteristic equation $F_K(x)$ defined by (4). **E** is the identity matrix.

$$\begin{aligned} F_K(x) &= \det(x\mathbf{E}-\mathbf{K}) = \det\begin{pmatrix} x-(k_1+k_2) & k_2 \\ k_2 & x-k_2 \end{pmatrix} \\ &= x^2 - (k_1+2k_2) + k_1k_2 \end{aligned} \tag{4}$$

The eigenvalues, α_1 and α_2, of **K** are the solutions for $F_K(x)=0$ and given by (5).

$$\alpha_1, \alpha_2 = \frac{(k_1+k_2) \pm \sqrt{k_1^2+4k_2^2}}{2} \tag{5}$$

If the values of k_1 and k_2 are positive, the determinant det**K** (=k_1k_2) given by (3) and the two eigenvalues α_1 and α_2 given by (5) become positive, suggesting that structural system is stable.

Similarly, stiffness matrix of a 3-story shear-beam model as shown in Figure 2(b), **K**, is given by (6).

$$\mathbf{K}_{3\times 3} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{6}$$

The determinant of $\mathbf{K}_{3\times 3}$ is calculated as (7).

$$\begin{aligned} &\det \mathbf{K}_{3\times 3} \\ &= (k_1+k_2)(k_2+k_3)k_3 - (-k_2)(-k_2)k_3 - (k_1+k_2)(-k_3)(-k_3) \\ &= k_1k_2k_3 \end{aligned} \tag{7}$$

Stiffness matrix of the n-story shear-beam model, $\mathbf{K}_{n\times n}$, is given by (8). It is well known that stiffness matrix of a shear-beam model is banded.

$$\mathbf{K}_{n\times n} = \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & O & \\ & -k_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & -k_{n-1} & \\ & O & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & & -k_n & k_n \end{bmatrix} \tag{8}$$

The determinant of $\mathbf{K}_{n\times n}$ can be found using the mathematical induction. Assume that $\det \mathbf{K}_{n\times n} = k_1k_2\cdots\cdots k_n \left(=\prod_{i=1}^{n} k_i\right)$ (9)

For n=2, 3, (9) is true as given by (3) and (7). Assumed that (9) is true for (n-1), then,

$$\det \mathbf{K}_{n-1\times n-1} = k_1k_2\cdots\cdots k_{n-1} \left(=\prod_{i=1}^{n-1} k_i\right) \tag{10}$$

The determinant of $\mathbf{K}_{n\times n}$ is related to $\mathbf{K}_{n-1\times n-1}$ as (11).

$$\begin{aligned} \det \mathbf{K}_{n\times n} &= \begin{vmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & O & \\ & -k_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & -k_{n-1} & \\ & O & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & & -k_n & k_n \end{vmatrix} \\ &= \left|\begin{array}{ccccc|c} k_1+k_2 & -k_2 & & & & 0 \\ -k_2 & k_2+k_3 & -k_3 & & O & \vdots \\ & -k_3 & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & -k_{n-1} & \vdots \\ & O & & -k_{n-1} & k_{n-1} & 0 \\ \hline & & & & -k_n & k_n \end{array}\right| \\ &= k_n \cdot \begin{vmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & -k_3 & \ddots & & \\ & & & k_{n-2}+k_{n-1} & -k_{n-1} \\ & & & -k_{n-1} & k_{n-1} \end{vmatrix} \\ &= k_n \cdot \det \mathbf{K}_{n-1\times n-1} \end{aligned} \tag{11}$$

As a result, the determinant of stiffness matrix of a n-story shear-beam model is given by (12).

$$\det \mathbf{K} = k_1k_2\cdots\cdots k_n \left(=\prod_{i=1}^{n} k_i\right) \tag{12}$$

This equation suggests that if any k_i becomes zero, the determinant of stiffness matrix becomes zero, then structural system becomes unstable.

The relation of the eigenvalues of $\mathbf{K}_{n\times n}$ and stiffness values of horizontal springs is investigated. The characteristic polynomial is given by (13). Here, $\alpha_1, \alpha_2 \cdots\cdots, \alpha_n$ are the eigenvalues of $\mathbf{K}_{n\times n}$. Letting $x=0$, $F_K(0)$ is expressed by both (14) and (15).

$$F_K(x) = \det(x\mathbf{E}_n - \mathbf{K}) = (x-\alpha_1)(x-\alpha_2)\cdots\cdots(x-\alpha_n) \tag{13}$$

$$\begin{cases} F_K(0) = \det(-\mathbf{K}) = (-1)^n \det \mathbf{K} \\ F_K(0) = (-1)^n \alpha_1\alpha_2\cdots\cdots\alpha_n \end{cases} \qquad (14), (15)$$

Therefore, (16) is derived.

$$\det \mathbf{K} = \alpha_1\alpha_2\cdots\cdots\alpha_n \tag{16}$$

From (12) and (16), (17) is derived.

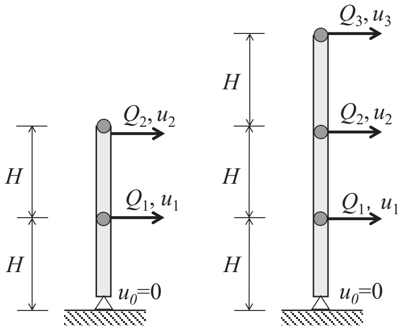
$$\begin{aligned} k_1k_2\cdots\cdots k_n &= \alpha_1\alpha_2\cdots\cdots\alpha_n \\ \prod_{i=1}^{n} k_i &= \prod_{i=1}^{n} \alpha_i \end{aligned} \tag{17}$$

Although it seems that $k_i = \alpha_i$, this is not true. This is since this is not satisfied for a 2-story shear-beam model as shown in (5).

2.2. FLEXURAL-BEAM MODEL

2.2.1. Flexural-beam supported by a pin

Stiffness matrices of 2-, 3-story flexural-beams supported by a pin at the basement as shown in Figure 3 are derived as follows.



(a) 2-story (b) 3-story

Figure 3. Flexural-beam supported by a pin

For one Bernoulli-Euler beam-element as shown in Figure 4, the relations of the forces, moments ${}^t\{Q_1, M_1, Q_2, M_2\}$ and the displacements, rotations ${}^t\{u_1, \theta_1, u_2, \theta_2\}$ are given by (18). Here, E is the elastic modulus, I is the moment of inertia and L is the length of the element.

$$\begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} \quad (18)$$

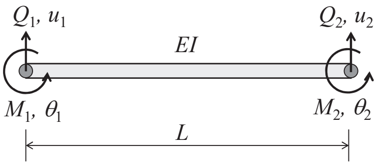


Figure 4. Beam element

2 beam-elements

The relations of the forces, moments and the displacements, rotations of 2 beam-elements connected to each other as shown in Figure 5 are obtained by assembling the stiffness matrices of 2 beam-elements, as given by (19). Here, for clarity, the following symbols are defined as

$\circledcirc = 12EL / L^3$, $\square = 6EI / L^2$, $\triangle = 2EI / L$.

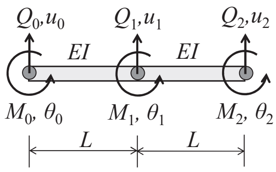


Figure 5. 3 beam-elements connected

$$\begin{Bmatrix} Q_0 \\ M_0 \\ Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \circledcirc & \square & -\circledcirc & \square & 0 & 0 \\ \square & 2\triangle & -\square & \triangle & 0 & 0 \\ -\circledcirc & -\square & 2\circledcirc & 0 & -\circledcirc & \square \\ \square & \triangle & 0 & 4\triangle & -\square & \triangle \\ 0 & 0 & -\circledcirc & -\square & \circledcirc & -\square \\ 0 & 0 & \square & \triangle & -\square & 2\triangle \end{bmatrix} \begin{Bmatrix} u_0 \\ \theta_0 \\ u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} \quad (19)$$

In order to conduct the static condensation and find the relations of forces and displacement using the condition of $\mathbf{M} = \mathbf{0}$, (19) is arranged to obtain (20), which can be expressed by (21).

$$\begin{Bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ M_0 \\ M_1 \\ M_2 \end{Bmatrix} = \left[\begin{array}{ccc:ccc} \circledcirc & -\circledcirc & 0 & \square & \square & 0 \\ -\circledcirc & 2\circledcirc & -\circledcirc & -\square & 0 & \square \\ 0 & -\circledcirc & \circledcirc & 0 & -\square & -\square \\ \hdashline \square & -\square & 0 & 2\triangle & \triangle & 0 \\ \square & 0 & -\square & \triangle & 4\triangle & \triangle \\ 0 & \square & -\square & 0 & \triangle & 2\triangle \end{array}\right] \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ \theta_0 \\ \theta_1 \\ \theta_2 \end{Bmatrix} \quad (20)$$

$$\begin{Bmatrix} \mathbf{Q} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} [\mathbf{K}_{11}] & [\mathbf{K}_{12}] \\ [\mathbf{K}_{21}] & [\mathbf{K}_{22}] \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\theta} \end{Bmatrix} \quad (21)$$

$$\mathbf{K}_{11} = \frac{12EI}{L^3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \mathbf{K}_{12} = \frac{6EI}{L^2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$
$$\mathbf{K}_{21} = \frac{6EI}{L^2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{K}_{22} = \frac{2EI}{L}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad (22)$$

Since $\mathbf{M} = \mathbf{0}$, (23) is derived.

$$\begin{aligned} &\mathbf{M} = \mathbf{K}_{21}\mathbf{u} + \mathbf{K}_{22}\boldsymbol{\theta} = \mathbf{0} \\ &\therefore \mathbf{K}_{22}\boldsymbol{\theta} = -\mathbf{K}_{21}\mathbf{u} \qquad (23) \\ &\therefore \boldsymbol{\theta} = -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{u} \end{aligned}$$

Substitute (23) into (21) and (24) is derived.

$$\begin{aligned} \mathbf{Q} &= \mathbf{K}_{11}\mathbf{u} + \mathbf{K}_{12}\boldsymbol{\theta} = \mathbf{K}_{11}\mathbf{u} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{u} \\ &= \left(\mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\right)\mathbf{u} \end{aligned} \quad (24)$$

$$\mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} = \frac{EI}{L^3}\begin{bmatrix} 1.5 & -3 & 1.5 \\ -3 & 6 & -3 \\ 1.5 & -3 & 1.5 \end{bmatrix} \quad (25)$$

As a result, the relations of the forces and displacements are derived as (26).

$$\begin{Bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 1.5 & -3 & 1.5 \\ -3 & 6 & -3 \\ 1.5 & -3 & 1.5 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} \quad (26)$$

Since $u_0 = 0$ as shown in Figure 3(a), letting $u_0 = 0$ in (26), the relations of the forces ${}^t\{Q_1, Q_2\}$ and displacements ${}^t\{u_1, u_2\}$ are derived as (27). Stiffness matrix $\mathbf{K_f}$ of the model as shown in Figure 3(a) is given by (28).

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{EI}{H^3}\begin{bmatrix} 6 & -3 \\ -3 & 1.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (27)$$

$$\mathbf{K_f} = \frac{EI}{H^3}\begin{bmatrix} 6 & -3 \\ -3 & 1.5 \end{bmatrix} \quad (28)$$

The determinant of $\mathbf{K_f}$ is calculated by (29) and is equal to zero suggesting that a flexural-beam supported by a pin at the basement is unstable itself.

$$\det \mathbf{K_f} = \left(\frac{EI}{H^3}\right)^2 \cdot \begin{vmatrix} 6 & -3 \\ -3 & 1.5 \end{vmatrix} = \left(\frac{EI}{H^3}\right)^2 \cdot \left(6 \cdot 1.5 - (-3)(-3)\right) = 0 \quad (29)$$

3 beam-elements

Similarly, the relations of the forces, moments and the displacements, rotations of 3 beam-elements connected to each other as shown in Figure 6 are obtained by assembling the stiffness matrices of 3 beam-elements, as given by (30).

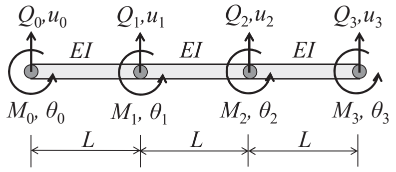


Figure 6. 3 beam-elements connected

$$\begin{Bmatrix} Q_0 \\ M_0 \\ Q_1 \\ M_1 \\ Q_2 \\ M_2 \\ Q_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} \circledcirc & \square & -\circledcirc & \square & 0 & 0 & 0 & 0 \\ \square & 2\triangle & -\square & \triangle & 0 & 0 & 0 & 0 \\ -\circledcirc & -\square & 2\circledcirc & 0 & -\circledcirc & \square & 0 & 0 \\ \square & \triangle & 0 & 4\triangle & -\square & \triangle & 0 & 0 \\ 0 & 0 & -\circledcirc & -\square & 2\circledcirc & 0 & -\circledcirc & \square \\ 0 & 0 & \square & \triangle & 0 & 4\triangle & -\square & \triangle \\ 0 & 0 & 0 & 0 & -\circledcirc & -\square & \circledcirc & -\square \\ 0 & 0 & 0 & 0 & \square & \triangle & -\square & 2\triangle \end{bmatrix} \begin{Bmatrix} u_0 \\ \theta_0 \\ u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} \tag{30}$$

Rearranging the terms according to the array of the forces, moments and the displacements and rotations, (31) is derived. This can be expressed by (32).

$$\begin{Bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ M_0 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \left[\begin{array}{cccc|cccc} \circledcirc & -\circledcirc & 0 & 0 & \square & \square & 0 & 0 \\ -\circledcirc & 2\circledcirc & -\circledcirc & 0 & -\square & 0 & \square & 0 \\ 0 & -\circledcirc & 2\circledcirc & -\circledcirc & 0 & -\square & 0 & \square \\ 0 & 0 & -\circledcirc & \circledcirc & 0 & 0 & -\square & -\square \\ \hline \square & -\square & 0 & 0 & 2\triangle & \triangle & 0 & 0 \\ \square & 0 & -\square & 0 & \triangle & 4\triangle & \triangle & 0 \\ 0 & \square & 0 & -\square & 0 & \triangle & 4\triangle & \triangle \\ 0 & 0 & \square & -\square & 0 & 0 & \triangle & 2\triangle \end{array}\right] \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \tag{31}$$

$$\begin{Bmatrix} \mathbf{Q} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} [\mathbf{K}_{11}] & [\mathbf{K}_{12}] \\ [\mathbf{K}_{21}] & [\mathbf{K}_{22}] \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\theta} \end{Bmatrix} \tag{32}$$

Here,

$$\mathbf{K}_{11} = \frac{12EI}{L^3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \mathbf{K}_{12} = \frac{6EI}{L^2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$
$$\mathbf{K}_{21} = \frac{6EI}{L^2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \mathbf{K}_{22} = \frac{2EI}{L} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{33}$$

Since $\mathbf{M} = \mathbf{0}$, (34) is derived.

$$\mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} = \frac{EI}{L^3} \begin{bmatrix} 1.6 & -3.6 & 2.4 & -0.4 \\ -3.6 & 9.6 & -8.4 & 2.4 \\ 2.4 & -8.4 & 9.6 & -3.6 \\ -0.4 & 2.4 & -3.6 & 1.6 \end{bmatrix} \tag{34}$$

Since $u_0 = 0$ in Figure 3(b), the relations of forces and displacements are given by (35) and then stiffness matrix $\mathbf{K_f}$ of the model as shown in Figure 3(b) is given by (36).

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \frac{EI}{H^3} \begin{bmatrix} 9.6 & -8.4 & 2.4 \\ -8.4 & 9.6 & -3.6 \\ 2.4 & -3.6 & 1.6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{35}$$

$$\mathbf{K_f} = \frac{EI}{H^3} \begin{bmatrix} 9.6 & -8.4 & 2.4 \\ -8.4 & 9.6 & -3.6 \\ 2.4 & -3.6 & 1.6 \end{bmatrix} \tag{36}$$

The determinant of $\mathbf{K_f}$ is calculated as (37) and is equal to zero suggesting that a flexural-beam supported by a pin at the basement is unstable itself.

$$\begin{aligned} \det \mathbf{K_f} &= \left(\frac{EI}{H^3}\right)^3 \cdot \begin{vmatrix} 9.6 & -8.4 & 2.4 \\ -8.4 & 9.6 & -3.6 \\ 2.4 & -3.6 & 1.6 \end{vmatrix} \\ &= \left(\frac{EI}{H^3}\right)^3 \cdot \begin{pmatrix} 9.6 \cdot 9.6 \cdot 1.6 + (-8.4) \cdot (-3.6) \cdot 2.4 \\ +2.4 \cdot (-8.4) \cdot (-3.6) - 2.4 \cdot 9.6 \cdot 2.4 \\ -(-8.4) \cdot (-8.4) \cdot 1.6 - 9.6 \cdot (-3.6) \cdot (-3.6) \end{pmatrix} \\ &= 0 \end{aligned} \tag{37}$$

2.2.2. Flexural-beam fully fixed at the basement

Stiffness matrices of 2-, 3-story flexural-beams fully fixed at the basement as shown in Figure 7 are derived as follows.

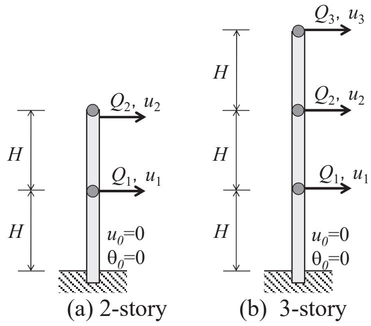


Figure 7. Flexural-beam fully fixed at the basement

2 beam-elements

The relations of the forces, moments and the displacements, rotations of 2 beam-elements connected to each other as shown in Figure 5 are obtained by assembling the stiffness matrices of 2 beam-elements, as given by (19). In a 2-story flexural-beam model as shown in Figure 7(a), since $u_0 = 0,\ \theta_0 = 0$, (38) is derived, which can be expressed by (39).

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ M_1 \\ M_2 \end{Bmatrix} = \left[\begin{array}{cc|cc} 2\circledcirc & -\circledcirc & 0 & \square \\ -\circledcirc & \circledcirc & -\square & -\square \\ \hline 0 & -\square & 4\triangle & \triangle \\ \square & -\square & \triangle & 2\triangle \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ \theta_1 \\ \theta_2 \end{Bmatrix} \tag{38}$$

$$\begin{Bmatrix} \mathbf{Q} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} [\mathbf{K}_{11}] & [\mathbf{K}_{12}] \\ [\mathbf{K}_{21}] & [\mathbf{K}_{22}] \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\theta} \end{Bmatrix} \tag{39}$$

Here,

$$\mathbf{K}_{11} = \frac{12EI}{L^3} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \mathbf{K}_{12} = \frac{6EI}{L^2} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$
$$\mathbf{K}_{21} = \frac{6EI}{L^2} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{K}_{22} = \frac{2EI}{L} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \tag{40}$$

Since $\mathbf{M} = \mathbf{0}$, stiffness matrix $\mathbf{K_f}$ of the model as shown in Figure 7(a) is derived as (41).

$$\mathbf{K_f} = [K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}] = \frac{EI}{H^3}\begin{bmatrix} \frac{96}{7} & -\frac{30}{7} \\ -\frac{30}{7} & \frac{12}{7} \end{bmatrix} \quad (41)$$

The determinant of $\mathbf{K_f}$ is calculated by (42) and this is positive suggesting that a flexural-beam fixed at the basement is stable.

$$\begin{aligned} \det \mathbf{K_f} &= \left(\frac{EI}{H^3}\right)^2 \cdot \begin{vmatrix} \frac{96}{7} & -\frac{30}{7} \\ -\frac{30}{7} & \frac{12}{7} \end{vmatrix} \\ &= \left(\frac{EI}{H^3}\right)^2 \cdot \left(\frac{1}{7}\right)^2 \left(96 \cdot 12 - (-30)\cdot(-30)\right) \\ &= \left(\frac{EI}{H^3}\right)^2 \cdot \left(\frac{1}{49}\right) \cdot 252 = \frac{36}{7}\left(\frac{EI}{H^3}\right)^2 \approx 5.143\left(\frac{EI}{H^3}\right)^2 \end{aligned} \quad (42)$$

3 beam-elements

Stiffness matrix of 3-story flexural-beam fully-fixed at the basement as shown in Figure 7(b) is derived as follows. Since $u_0 = 0,\ \theta_0 = 0$, the relations are derived as (43).

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \left[\begin{array}{ccc:ccc} 2\circledcirc & -\circledcirc & 0 & 0 & \square & 0 \\ -\circledcirc & 2\circledcirc & -\circledcirc & -\square & 0 & \square \\ 0 & -\circledcirc & \circledcirc & 0 & -\square & -\square \\ \hdashline 0 & -\square & 0 & 4\triangle & \triangle & 0 \\ \square & 0 & -\square & \triangle & 4\triangle & \triangle \\ 0 & \square & -\square & 0 & \triangle & 2\triangle \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \quad (43)$$

$\mathbf{K}_{11}$, $\mathbf{K}_{12}$, $\mathbf{K}_{21}$, $\mathbf{K}_{22}$ in the form of (39) are as follows.

$$\mathbf{K}_{11} = \frac{12EI}{L^3}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \mathbf{K}_{12} = \frac{6EI}{L^2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$
$$\mathbf{K}_{21} = \frac{6EI}{L^2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{K}_{22} = \frac{2EI}{L}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad (44)$$

Stiffness matrix $\mathbf{K_f}$ is derived as (45).

$$\mathbf{K_f} = [K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}] = \frac{EI}{H^3}\begin{bmatrix} \frac{240}{13} & -\frac{138}{13} & \frac{36}{13} \\ -\frac{138}{13} & \frac{132}{13} & -\frac{48}{13} \\ \frac{36}{13} & -\frac{48}{13} & \frac{21}{13} \end{bmatrix}$$

(45)

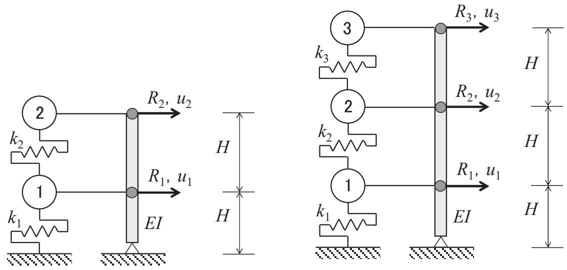
The determinant of $\mathbf{K_f}$ is calculated as (46) and this is positive.

$$\begin{aligned} \det \mathbf{K_f} &= \left(\frac{EI}{H^3}\right)^3 \cdot \begin{vmatrix} \frac{240}{13} & -\frac{138}{13} & \frac{36}{13} \\ -\frac{138}{13} & \frac{132}{13} & -\frac{48}{13} \\ \frac{36}{13} & -\frac{48}{13} & \frac{21}{13} \end{vmatrix} \\ &= \left(\frac{EI}{H^3}\right)^3 \cdot \left(\frac{1}{13}\right)^3 \begin{pmatrix} 240\cdot 132 \cdot 21 + (-138)\cdot(-48)\cdot 36 \\ +36\cdot(-138)\cdot(-48) - 36\cdot 132\cdot 36 \\ -(-138)\cdot(-138)\cdot 21 - 240\cdot(-48)\cdot(-48) \end{pmatrix} \\ &= \left(\frac{EI}{H^3}\right)^3 \cdot \left(\frac{1}{13}\right)^3 \cdot 18252 = \frac{108}{13}\left(\frac{EI}{H^3}\right)^3 \approx 8.3077\left(\frac{EI}{H^3}\right)^3 \end{aligned}$$

(46)

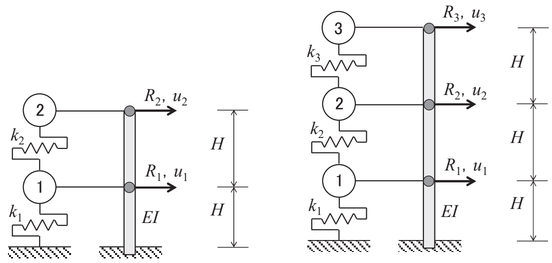
2.3. COUPLED SHEAR-FLEXURAL-BEAM MODEL

Stiffness matrices of the 2- and 3-story coupled shear-flexural-beam models as shown in Figures 8 and 9 are derived as follows. The shear-beam and flexural-beam are connected to each other with rigid link at each floor level.



(a) 2-story (b) 3-story

Figure 8. Coupled shear-flexural-beam models (pin-supported)



(a) 2-story (b) 3-story

Figure 9. Coupled shear-flexural-beam models (fully-fixed)

The displacements at the same floor level of the shear-beam and flexural-beam in the coupled shear-flexural-beam model are identical and then (47) is derived. The forces of the coupled shear-flexural-beam model is a sum of forces of the shear-beam and forces of the flexural-beam and then (48) is derived.

$$u_1 = u_{s1} = u_{f1}, \quad u_2 = u_{s2} = u_{f2}, \quad u_3 = u_{s3} = u_{f3} \quad (47)$$

$$R_1 = R_{s1} + R_{f1}, \quad R_2 = R_{s2} + R_{f2}, \quad R_3 = R_{s3} + R_{f3} \quad (48)$$

The relations of the force and displacements of 2-story coupled shear-flexural-beam model with a pin-supported flexural-beam is derived as (49). Therefore, stiffness matrix is given by (50).

$$\begin{aligned} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} &= \begin{pmatrix} R_{s1} \\ R_{s2} \end{pmatrix} + \begin{pmatrix} R_{f1} \\ R_{f2} \end{pmatrix} \\ &= \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_{s1} \\ u_{s2} \end{Bmatrix} + \frac{EI}{H^3}\begin{bmatrix} 6 & -3 \\ -3 & 1.5 \end{bmatrix}\begin{Bmatrix} u_{f1} \\ u_{f2} \end{Bmatrix} \\ &= \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{EI}{H^3}\begin{bmatrix} 6 & -3 \\ -3 & 1.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ &= \left(\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} + \frac{EI}{H^3}\begin{bmatrix} 6 & -3 \\ -3 & 1.5 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \end{aligned} \quad (49)$$

$$\mathbf{K} = \mathbf{K_s} + \mathbf{K_f} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} + \frac{EI}{H^3}\begin{bmatrix} 6 & -3 \\ -3 & 1.5 \end{bmatrix} \quad (50)$$

Similarly, stiffness matrix of 3-story coupled shear-flexural-beam model with a pin-supported flexural-beam is derived as (51).

$$\mathbf{K}=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}+\frac{EI}{H^3}\begin{bmatrix} 9.6 & -8.4 & 2.4 \\ -8.4 & 9.6 & -3.6 \\ 2.4 & -3.6 & 1.6 \end{bmatrix} \quad (51)$$

When the flexural-beam is fully fixed at the basement, stiffness matrices of 2-, 3-story coupled shear-flexural-beam models are given by (52), (53), respectively.

$$\mathbf{K}=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}+\frac{EI}{H^3}\begin{bmatrix} \frac{96}{7} & -\frac{30}{7} \\ -\frac{30}{7} & \frac{12}{7} \end{bmatrix} \quad (52)$$

$$\mathbf{K}=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}+\frac{EI}{H^3}\begin{bmatrix} \frac{240}{13} & -\frac{138}{13} & \frac{36}{13} \\ -\frac{138}{13} & \frac{132}{13} & -\frac{48}{13} \\ \frac{36}{13} & -\frac{48}{13} & \frac{21}{13} \end{bmatrix} \quad (53)$$

3. Static stability of coupled shear-flexural-beam models

In order to evaluate static stability of the coupled shear-flexural-beam model, eigenvalue analyses are conducted. Eigenvalue problem is solved using (54). Here, Ω_i is the i^{th}-mode eigenvalue of stiffness matrix **K** normalized by mass matrix **M**, which is equal to a square of the i^{th}-mode natural frequency, ω_i. The i^{th}-mode natural period T_i is calculated as (55).

$$\det(\mathbf{K}-\Omega_i\mathbf{M})=0 \quad \text{where, } \Omega_i=\omega_i^2 \quad (54)$$

$$T_i=\frac{2\pi}{\omega_i} \quad (55)$$

The stiffness of each story in the shear-beam model is calculated using (56), which provides uniform distribution of the story drift angles for specified horizontal forces added at i^{th}-floor, f_i. Here, m_i is i-story mass, h_i is i^{th}-floor height. The 1st-natural periods, T, are set to 0.24 sec and 0.36 sec for 2- and 3-story models, respectively.

$$k_i=\frac{4\pi^2}{T^2}\frac{\sum_{i=1}^{n} m_i h_i^2}{\sum_{i=1}^{n} f_i h_i}\frac{\sum_{j=i}^{n} f_j}{(h_{i+1}-h_i)} \quad (56)$$

Each story stiffness of 2- and 3-story shear-beam model is calculated as (57), (58), respectively, assuming that all masses are equal to m, each story height is equal to H, and horizontal forces added at i^{th}-floor, f_i are an inverted triangle distribution.

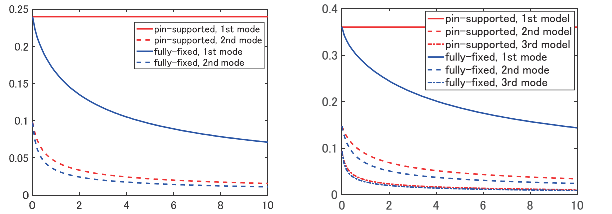
$$k_1=\frac{4\pi^2}{T^2}\frac{m(H^2+4H^2)}{(fH+4fH)}\frac{(f+2f)}{H}=\frac{12m\pi^2}{T^2},\quad k_2=\frac{8m\pi^2}{T^2} \quad (57)$$

$$k_1=\frac{4\pi^2}{T^2}\frac{m(H^2+4H^2+9H^2)}{(fH+4fH+9fH)}\frac{(f+2f+3f)}{H}=\frac{24m\pi^2}{T^2}$$
$$k_2=\frac{20m\pi^2}{T^2},\quad k_3=\frac{12m\pi^2}{T^2} \quad (58)$$

The natural periods of 2- and 3-story models are calculated for various flexural-stiffness ratio, α_{cc}, defined by (59). Here, EI is the flexural stiffness of the flexural-beam, H is a story height and k_1 is the initial stiffness of the 1st-story in the shear-beam.

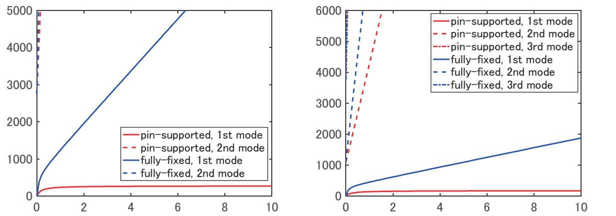
$$\alpha_{cc}=\frac{EI/H^3}{k_1} \quad (59)$$

The natural periods for all modes for 2- and 3-story coupled shear-flexural-beam models with a pin-supported or fully-fixed flexural-beam are plotted in Figure 10. As α_{cc} increases, the 1st-mode natural period of the model with a pin-supported flexural-beam does not decrease since the pin-supported flexural-beam rotates rigidly. However, higher-mode natural periods decrease. For the model with a fully-fixed flexural-beam, the natural periods of all modes decrease. Therefore, a pin-supported flexural-beam increases the stability for all modes except the 1st-mode, and a fully-fixed flexural-beam increases the stability for all modes.



(x-axis:α_{cc}, y-axis: natural period (sec), red: pin-supported, blue: fixed)
(a) 2-story (b) 3-story
Figure 10. Natural periods (at elastic)

The eigenvalue analyses are conducted for 2- and 3-story models with assumed 1st-story mechanism, setting the 1st-story tangent stiffness of the shear-beam to zero. The instantaneous eigenvalues are plotted for various α_{cc} in Figure 11. As α_{cc} increases, the eigenvalues of all modes increase, suggesting that the flexural-beam increases the stability of entire structure under the story-mechanism in a particular story, activating the C.C. effects.



(x-axis:α_{cc}, y-axis: Ω_i (1/sec^2) , red: pin-supported, blue: fixed)
(a) 2-story (b) 3-story
Figure 11. Instantaneous eigenvalues (at 1st-story mechanism)

4. Conclusions

The stiffness matrices of a shear-beam, flexural-beam, and the coupled shear-flexural-beam model derived. Eigenvalue analyses are conducted for 2- and 3-story coupled shear-flexural-beam models at elastic state or assumed 1st-story mechanism to demonstrate the C.C. effects on static stability of entire structure.

The coupled shear-flexural-beam model used in this study can simulate the story-failure, which was observed for piloti-type RC buildings and wooden houses in the 1995 Hyogoken-Nanbu earthquake. The pancake collapse, which was typically observed in the 2023 Turkey-Syria earthquake, is similar to the story-failure, accompanied by progressive collapse from the top to the bottom.

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